Image restoration: constrained approaches
— Support and positivity —

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Topics

- Image restoration, deconvolution
  - Motivating examples: medical, astrophysical, industrial, vision, . . .
  - Various problems: deconvolution, Fourier synthesis, denoising. . .
  - Missing information: ill-posed character and regularisation

- Three types of regularised inversion
  1. Quadratic penalties and linear solutions
    - Closed-form expression
    - Computation through FFT
    - Optimisation (e.g., gradient), system solvers (e.g., splitting)
  2. Non-quadratic penalties and edge preservation
    - Half-quadratic approaches, including computation through FFT
    - Optimisation (e.g., gradient), system solvers (e.g., splitting)
  3. Constraints: positivity and support
    - Augmented Lagrangian and ADMM, including computation by FFT
    - Optimisation (e.g., gradient), system solvers (e.g., splitting)

- Bayesian strategy: a few incursions
  - Tuning hyperparameters, instrument parameters, . . .
  - Hidden / latent parameters, segmentation, detection, . . .
Convolution / Deconvolution

\[ y = Hx + \varepsilon = h \ast x + \varepsilon \]

\[ \hat{x} = \hat{X}(y) \]

**Restoration, deconvolution-denoising**

- General problem: ill-posed inverse problems, *i.e.*, lack of information
- Methodology: regularisation, *i.e.*, information compensation
  - Specificity of the inversion / reconstruction / restoration methods
  - Trade off and tuning parameters
- Limited quality results
Known: $H$ and $y$ / Unknown: $x$

Compare observations $y$ and model output $Hx$

$$J_{LS}(x) = \|y - Hx\|^2$$

Quadratic penalty of the gray level gradient (or other linear combinations)

$$\mathcal{P}(x) = \sum_{p \sim q} (x_p - x_q)^2 = \|Dx\|^2$$

Least squares and quadratic penalty:

$$J_{PLS}(x) = \|y - Hx\|^2 + \mu \|Dx\|^2$$
Quadratic penalty: criterion and solution

- Least squares and quadratic penalty:
  \[
  J_{\text{PLS}}(x) = \| y - Hx \|^2 + \mu \| Dx \|^2
  \]

- Restored image
  \[
  \hat{x}_{\text{PLS}} = \arg \min_x J_{\text{PLS}}(x)
  \]
  \[
  (H^tH + \mu D^tD) \hat{x}_{\text{PLS}} = H^ty
  \]
  \[
  \hat{x}_{\text{PLS}} = (H^tH + \mu D^tD)^{-1} H^ty
  \]

- Computations based on diagonalization through FFT
  \[
  \hat{\mathbf{x}} = (\Lambda^\dagger_h \Lambda_h + \mu \Lambda^\dagger_d \Lambda_d)^{-1} \Lambda^\dagger_h \hat{\mathbf{y}}
  \]
  \[
  \hat{x}_n = \frac{\hat{h}^*}{|\hat{h}_n|^2 + \mu |\hat{d}_n|^2} \hat{y}_n \quad \text{for } n = 1, \ldots N
  \]
## Various options and many relationships

- Direct calculus, compact (closed) form, matrix inversion
- Algorithms for linear system
  - Gauss, Gauss-Jordan
  - Substitution
  - Triangularisation,
- Numerical optimisation
  - gradient descent... and various modifications
  - Pixel wise, pixel by pixel
- Diagonalization
  - Circulant approximation and diagonalization by FFT
- Special algorithms, especially for 1D case
  - Recursive least squares
  - Kalman smoother or filter (and fast versions,...)
Solution from least squares and quadratic penalty
Synthesis and extensions to constraints

- Limited capability to manage conflict between
  - Smoothing and
  - Avoiding noise explosion
  ... that limits resolution capabilities

Extension to non-quadratic penalty
- Less “smoothing” around “discontinuities”
  - Ambivalence:
    - Smoothing (homogeneous regions)
    - Heightening, enhancement, sharpening (discontinuities, edges)
  ... and new compromise, trade off, conciliation

Another extension: include constraints
- Positivity and support
- Better physics and improved resolution
- Resort to the linear solution and FFT (Wiener-Hunt)
  - Augmented Lagrangian and ADMM
Taking constraints into account

- **Expected benefits**
  - Better physical modelling
  - More information $\leadsto$ “quality” improvement
  - Improved resolution

- **Restoration technology**
  - Still based on a penalised criterion...
    
    $$J_{\text{PLS}}(x) = \|y - Hx\|^2 + \mu \|Dx\|^2$$

  - ...restored image still defined as a minimiser...
    
    $$\hat{x} = \arg \min_x J_{\text{PLS}}(x)$$

  - ...but including constraints
    
    ...(about the value of the gray level of pixels)
Taking constraints into account: positivity and support

- **Notation**
  - \( \mathcal{M} \): index set of the image pixels
  - \( S, D \): index set of a subset (support, region, mask, …) of the pixels

- **Investigated constraints here**
  - **Positivity**
    \[ C_p : \forall p \in \mathcal{M}, \quad x_p \geq 0 \]
  - **Support, mask**
    \[ C_s : \forall p \in \bar{S}, \quad x_p = 0 \]

- **Extensions (non investigated here)**
  - **Template**
    \[ \forall p \in \mathcal{M}, \quad t_p^- \leq x_p \leq t_p^+ \]
  - **Partially known map**
    \[ \forall p \in D, \quad x_p = m_p \]
Taking constraints into account: positivity and support

**General form inequality / equality**

\[ Bx - b \geq 0 \quad \text{et} \quad Ax - a = 0 \]

- **Positivity**
  \[ C_p : \forall p \in \mathcal{M}, \quad x_p \geq 0 \quad \rightsquigarrow \quad B = I \quad \text{et} \quad b = 0 \]

- **Support**
  \[ C_s : \forall p \in \bar{S}, \quad x_p = 0 \quad \rightsquigarrow \quad A = T_S \quad \text{et} \quad a = 0 \]

- **Template**
  \[ \forall p \in \mathcal{M}, \quad t_p^- \leq x_p \quad \rightsquigarrow \quad B = I \quad \text{et} \quad b = t^- \]
  \[ x_p \leq t_p^+ \quad \rightsquigarrow \quad B = -I \quad \text{et} \quad b = -t^+ \]

- **Partially known map**
  \[ \forall p \in \mathcal{D}, \quad x_p = m_p \quad \rightsquigarrow \quad A = T_D \quad \text{et} \quad a = m \]
Constrained minimiser

Theoretical point: criterion, constraint and property

- Quadratic criterion: \( J_{\text{PLS}}(x) = \|y - Hx\|^2 + \mu \|Dx\|^2 \)

- Linear constraints:
  \[
  \begin{align*}
  x_p &= 0 \quad \text{for } p \in \bar{S} \\
  x_p &\geq 0 \quad \text{for } p \in \mathcal{M}
  \end{align*}
  \]

- Question of convexity
  - Convex (strict) criterion
  - Convex constraint set

Theoretical point: construction of the solution

- Solution: *the only constrained minimiser*
  \[
  \hat{x} = \arg \min_x \left\{ \|y - Hx\|^2 + \mu \|Dx\|^2 \right\}
  \]
  \[
  \text{s.t.} \quad \begin{align*}
  x_p &= 0 \quad \text{for } p \in \bar{S} \\
  x_p &\geq 0 \quad \text{for } p \in \mathcal{M}
  \end{align*}
  \]
Constraints: some illustrations
One variable: $\alpha(t - \bar{t})^2 + \gamma$

- Unconstrained solution: $\hat{t} = \bar{t}$
- Constrained solution: $\hat{t} = \max[0, \bar{t}]$

Active and inactive constraints
Positivity: two variables (1)

- Two variables: $\alpha_1(t_1 - \bar{t}_1)^2 + \alpha_2(t_2 - \bar{t}_2)^2 + \beta(t_2 - t_1)^2 + \gamma$

- Sometimes / often difficult to deduce
  - the constrained minimiser
  - from the unconstrained one
Positivity: two variables (2)

- Two variables: $\alpha_1(t_1 - \bar{t}_1)^2 + \alpha_2(t_2 - \bar{t}_2)^2 + \beta(t_2 - t_1)^2 + \gamma$

- Constrained solution = Unconstrained solution (1)
- Constrained solution $\neq$ Unconstrained solution (2)
  
  ... so active constraints
Positivity: two variables (3)

- Two variables: \( \alpha_1(t_1 - \bar{t}_1)^2 + \alpha_2(t_2 - \bar{t}_2)^2 + \beta(t_2 - t_1)^2 + \gamma \)

![Graph 2a](image1)

- Constrained solution \(\neq\) Unconstrained solution (2)
  - . . . so active constraints
    - Constrained solution \(\neq\) Projected unconstrained solution (2a)
      \((\hat{t}_1; \hat{t}_2) \neq (\max [0, \bar{t}_1]; \max [0, \bar{t}_2])\)
    - Constrained solution = Projected unconstrained solution (2b)
      \((\hat{t}_1; \hat{t}_2) = (\max [0, \bar{t}_1]; \max [0, \bar{t}_2])\)
Problem

- Quadratic optimisation with linear constraints
- Difficulties
  - \( N \sim 1\,000\,000 \)
  - Constraints ⊕ non-separable variables

Existing algorithms

- Existing tools *with guaranteed convergence*
  
  [Bertsekas 95,99; Nocedal 00,08; Boyd 04,11]

- Gradient projection methods, constrained gradient method
- Broyden-Fletcher-Goldfarb-Shanno (BFGS) and limited memory
- Interior points and barrier
- Pixel-wise descent
- Augmented Lagrangian, ADMM
  - Constrained but separated + non-separated but non-constrained
  - Partial solutions still through FFT
Equality constraints

\[ \hat{x} = \arg \min_x \left\{ \| y - Hx \|^2 + \mu \| Dx \|^2 \right\} \]
\[ \text{s.t. } x_p = 0 \quad \text{for } p \in \bar{S} \]

- **Sets and subsets of pixels**
  - \( M \): full vector of pixels \( \sim x \in \mathbb{R}^N \)
  - \( S \): vector of unconstrained pixels \( \sim \bar{x} \in \mathbb{R}^M \)

- **Truncation**
  - \( \bar{x} = T x \) truncation, selection of unconstrained pixels
  - \( T \) is \( M \times N \) (\( M < N \)), e.g., \( T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \)

- **Properties: zero-padding,\ldots**
  - \( T^t \bar{x} \) zero-padding, fill with zeros
  - \( TT^t = I_M \)
  - \( T^t T = \text{diag}[ \ldots 0 / 1 \ldots ] \): projection, “nullification matrix”
Equality: direct closed form expression

- Original (unconstrained) criterion
  \[ J_{\text{PLS}}(x) = \| y - Hx \|^2 + \mu \| Dx \|^2 \]

- Zero-padded variable
  \[ x = T^t \bar{x} \]

- Restricted criterion
  \[ \bar{J}_{\text{PLS}}(\bar{x}) = \| y - HT^t \bar{x} \|^2 + \mu \| DT^t \bar{x} \|^2 \]

- Closed form expression for the solution
  \[
  \hat{\bar{x}} = \arg \min_{\bar{x} \in \mathbb{R}^M} \bar{J}_{\text{PLS}}(\bar{x}) \\
  = \left[ TH^t HT^t + \mu TD^t DT^t \right]^{-1} TH^t y \\
  = \left[ T (H^t H + \mu D^t D) T^t \right]^{-1} TH^t y \\
  \]
  \[
  \hat{x} = T^t \bar{x} \\
  = T^t \left[ T (H^t H + \mu D^t D) T^t \right]^{-1} TH^t y
  \]
Equality: closed form expression via Lagrangian

- Original (unconstrained) criterion
  \[ J_{\text{PLS}}(x) = \| y - Hx \|^2 + \mu \| Dx \|^2 \]

- Equality constraints:
  \[
  \begin{align*}
  x_p &= 0 \text{ for } p \in \bar{S} \\
  \bar{T}x &= 0
  \end{align*}
  \]

- Equality constraints and Lagrangian term
  \[
  \sum_{p \in \bar{S}} \ell_p x_p = \ell^t \bar{T}x
  \]

- Lagrangian
  \[
  \mathcal{L}(x, \ell) = \| y - Hx \|^2 + \mu \| Dx \|^2 + \ell^t \bar{T}x
  \]

- Closed form expression (see exercise)
  \[
  \hat{x} = \left[ Q^{-1} - Q^{-1} \bar{T}^t (\bar{T}Q^{-1} \bar{T}^t)^{-1} \bar{T}Q^{-1} \right] H^t y
  \]
  \[
  Q = (H^t H + \mu D^t D)
  \]
Original (unconstrained) criterion

\[ J_{\text{PLS}}(x) = \|y - Hx\|^2 + \mu \|Dx\|^2 \]

Equality constraints:

\[ \bar{T}x = 0 \]

Lagrangian

\[ L(x, \ell) = \|y - Hx\|^2 + \mu \|Dx\|^2 + \ell^t\bar{T}x \]

Iterative algorithm

\[
\begin{cases}
    x^{[k+1]} = \arg \min_x L(x, \ell^{[k]}) = (H^tH + \mu D^tD)^{-1}(H^ty - \bullet) \\
    \ell^{[k+1]} = \ell^{[k]} + \tau_k \bar{T}x^{[k+1]}
\end{cases}
\]
Equality: practical algorithm via Lagrangian

- Original (unconstrained) criterion
  \[ \mathcal{J}_{\text{PLS}}(x) = \|y - Hx\|^2 + \mu \|Dx\|^2 \]

- Equality constraints:
  \[ \bar{T}x = 0 \]

- Lagrangian
  \[ \mathcal{L}(x, \ell) = \|y - Hx\|^2 + \mu \|Dx\|^2 + \ell^t \bar{T}x \]

- Iterative algorithm
  \[
  \begin{align*}
  x^{[k+1]} &= \arg \min_x \mathcal{L}(x, \ell^{[k]}) = (H^t H + \mu D^t D)^{-1} (H^t y - \bar{T}^t \ell^{[k]}/2) \\
  \ell^{[k+1]} &= \ell^{[k]} + \tau_k \bar{T}x^{[k+1]}
  \end{align*}
  \]
Equality: algorithm via augmented Lagrangian

- Original (unconstrained) criterion
  \[ J_{\text{PLS}}^+(x) = \| y - Hx \|^2 + \mu \| Dx \|^2 + \rho \| \bar{T}x \|^2 \]

- Equality constraints:
  \[ \bar{T}x = 0 \]

- Lagrangian
  \[ \mathcal{L}_\rho(x, \ell) = \| y - Hx \|^2 + \mu \| Dx \|^2 + \rho \| \bar{T}x \|^2 + \ell^t \bar{T}x \]

- Iterative algorithm
  \[
  \begin{align*}
  x^{[k+1]} &= (H^t H + \mu D^t D + \cdot)^{-1}(H^t y - \bar{T}^t \ell^{[k]} / 2) \\
  \ell^{[k+1]} &= \ell^{[k]} + 2\rho \bar{T}x^{[k+1]}
  \end{align*}
  \]
Original (unconstrained) criterion

\[ J_{\text{PLS}}^+(x) = \| y - Hx \|^2 + \mu \| Dx \|^2 + \rho \| \bar{T}x \|^2 \]

Equality constraints:

\[ \bar{T}x = 0 \]

Lagrangian

\[ \mathcal{L}_\rho(x, \ell) = \| y - Hx \|^2 + \mu \| Dx \|^2 + \rho \| \bar{T}x \|^2 + \ell^T \bar{T}x \]

Iterative algorithm

\[
\begin{align*}
    x^{[k+1]} &= (H^t H + \mu D^t D + \rho T^t T)^{-1} (H^t y - \bar{T}^t \ell^{[k]} / 2) \\
    \ell^{[k+1]} &= \ell^{[k]} + 2\rho \bar{T}x^{[k+1]}
\end{align*}
\]
Equality: via augmented Lagrangian and slack variables

- Original (unconstrained) criterion

\[ J_{\text{PLS}}(x) = \| y - Hx \|^2 + \mu \| Dx \|^2 \]

- Constraint \( \oplus \) auxiliary (slack) variables

\[ x_p = 0 \text{ for } p \in \bar{S} \quad \implies \quad \begin{cases} x_p = s_p & \text{for } p \in M \\ s_p = 0 & \text{for } p \in \bar{S} \end{cases} \]

- Augmented Lagrangian \( \oplus \) slack variables

\[ \mathcal{L}_\rho(x, s, \ell) = \| y - Hx \|^2 + \mu \| Dx \|^2 + \rho \| x - s \|^2 + \ell^t(x - s) \]

- Iterative algorithm

\[
\begin{align*}
    x^{[k+1]} &= (H^tH + \mu D^tD + \rho I)^{-1}(H^ty - \ell^{[k]}/2 + \bullet) \\
    s_p^{[k+1]} &= \begin{cases} 
        \bullet & \text{for } p \in S \\
        0 & \text{for } p \in \bar{S} 
    \end{cases} \\
    \ell^{[k+1]} &= \ell^{[k]} + 2\rho \left( x^{[k+1]} - s^{[k+1]} \right)
\end{align*}
\]
Equality: via augmented Lagrangian and slack variables

- **Original (unconstrained) criterion**
  \[
  J_{\text{PLS}}(x) = \| y - Hx \|^2 + \mu \| Dx \|^2
  \]

- **Constraint + auxiliary (slack) variables**
  \[
  x_p = 0 \quad \text{for } p \in \bar{S} \quad \implies \quad \begin{cases} 
  x_p = s_p & \text{for } p \in M \\
  s_p = 0 & \text{for } p \in \bar{S}
  \end{cases}
  \]

- **Augmented Lagrangian + slack variables**
  \[
  L_\rho(x, s, \ell) = \| y - Hx \|^2 + \mu \| Dx \|^2 + \rho \| x - s \|^2 + \ell^t (x - s)
  \]

- **Iterative algorithm**
  \[
  \begin{aligned}
  &x^{[k+1]} = (H^t H + \mu D^t D + \rho I)^{-1} (H^t y - \ell^{[k]}/2 + \rho s^{[k]}) \\
  &s_p^{[k+1]} = \begin{cases} 
  \bullet & \text{for } p \in S \\
  0 & \text{for } p \in \bar{S}
  \end{cases} \\
  &\ell^{[k+1]} = \ell^{[k]} + 2\rho \left( x^{[k+1]} - s^{[k+1]} \right)
  \end{aligned}
  \]
Equality: via augmented Lagrangian and slack variables

- Original (unconstrained) criterion
  \[ J_{\text{PLS}}(x) = \|y - Hx\|^2 + \mu \|Dx\|^2 \]

- Constraint ⊕ auxiliary (slack) variables
  \[ x_p = 0 \text{ for } p \in \bar{S} \quad \leadsto \quad \begin{cases} x_p = s_p & \text{for } p \in M \\ s_p = 0 & \text{for } p \in \bar{S} \end{cases} \]

- Augmented Lagrangian ⊕ slack variables
  \[ L_\rho(x, s, \ell) = \|y - Hx\|^2 + \mu \|Dx\|^2 + \rho \|x - s\|^2 + \ell^t(x - s) \]

- Iterative algorithm
  \[
  \begin{align*}
  x^{[k+1]} &= (H^t H + \mu D^t D + \rho I)^{-1}(H^t y - \ell^{[k]}/2 + \rho s^{[k]}) \\
  s_p^{[k+1]} &= \begin{cases} 
  x_p^{[k+1]} + \ell_p^{[k]}/(2\rho) & \text{for } p \in S \\
  0 & \text{for } p \in \bar{S} 
  \end{cases} \\
  \ell^{[k+1]} &= \ell^{[k]} + 2\rho \left( x^{[k+1]} - s^{[k+1]} \right)
  \end{align*}
  \]
Equality and inequality constraints: problem

- Original (unconstrained) criterion
  \[ \mathcal{J}_{\text{PLS}}(x) = \| y - Hx \|^2 + \mu \| Dx \|^2 \]

- Equality and inequality constraints
  \[
  \begin{align*}
  x_p &= 0 \quad \text{for } p \in \bar{S} \\
  x_p &\geq 0 \quad \text{for } p \in M
  \end{align*}
  \]

- Equality and inequality constraints \( \oplus \) slack variables
  \[
  \begin{align*}
  x_p &= s_p \quad \text{for } p \in M \\
  s_p &= 0 \quad \text{for } p \in \bar{S} \\
  s_p &\geq 0 \quad \text{for } p \in M
  \end{align*}
  \]

- Augmented Lagrangian \( \oplus \) slack variables
  \[
  \mathcal{L}_\rho(x, s, \ell) = \| y - Hx \|^2 + \mu \| Dx \|^2 + \rho \| x - s \|^2 + \ell^t(x - s)
  \]
Iterative algorithm: ADMM

\[ \mathcal{L}(x, s, \ell) = \|y - Hx\|^2 + \mu \|Dx\|^2 + \rho \|x - s\|^2 + \ell^t(x - s) \]

- Iterate three steps

1. Unconstrained minimisation w.r.t. \( x \)
   \[ \tilde{x} = (H^tH + \mu D^tD + \rho I)^{-1} (H^ty + [\rho s - \ell/2]) \quad (\equiv \text{FFT}) \]

2. Constrained minimisation w.r.t. \( s \) (s.t. \( s_p \geq 0 \) or \( s_p = 0 \))
   \[ \tilde{s}_p = \begin{cases} 
   \max(0, x_p + \ell_p/(2\rho)) & \text{for } p \in S \\
   0 & \text{for } p \in \bar{S} 
   \end{cases} \]

3. Update \( \ell \)
   \[ \tilde{\ell}_p = \ell_p + 2\rho(x_p - s_p) \]
### Object update: other possibilities

Various options and many relationship...

- Direct calculus, closed-form expression, matrix inversion
- Algorithm for linear systems
  - Gauss, Gauss-Jordan
  - Substitution
  - Triangularisation, ...
- Numerical optimisation
  - Gradient descent. . . and modified versions
  - Pixel wise, pixel by pixel
- Diagonalization
  - Circulant approximation and diagonalization by FFT
- Special algorithms, especially for 1D case
  - Recursive least squares
  - Kalman smoother or filter (and fast versions)
Constrained solution
## Conclusions

### Synthesis

- Image deconvolution
- Taking constraints into account
  - Positivity and support
  - Quadratic penalty
- Numerical computations: augmented Lagrangian and ADMM
  - Iterative: quadratic ⊕ separable
    - Circulant case (diagonalization) \(\leadsto\) FFT only
    - (or numerical optimisation, system solvers, . . .)
  - Parallel (separable and explicit)

### Extensions (not developed)

- Also available for
  - non-invariant linear direct model
  - colour images, multispectral and hyperspectral
  - also signal, 3D and more, video, 3D+t . . .
- Including both Huber penalty and constraints
- Hyperparameters estimation, instrument parameter estimation, . . .