Image restoration
— Convex approaches: penalties and constraints —

An example in astronomy

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Summary

- Direct model and inverse problem
  - Interpolation-extrapolation / deconvolution / Fourier synthesis
  - Indetermination, non-inversibility

- Prior information and regularized solution
  - Positivity and possible support
  - Point sources onto smooth background and double model

- Algorithmic aspects and numerical optimisation

- Data processing results
  - Simulated Data
  - NRH Data

- Conclusions et extensions
Interferometry: principles of measurement

Physical principle [Thompson, Moran, Swenson, 2001]

- Antenna array \(\leadsto\) large aperture
- Frequency band, e.g., 164 MHz
- Couple of antennas interference \(\leadsto\) one measure in the Fourier plane

**Picture site (NRH)**

Knowledge of the sun, magnetic activity, eruptions, sunspots, ...  
Forecast of sun events and their impact, ...
Interferometer: example of measurements

\[ x \quad Fx \quad TFx \quad y = TFx + \varepsilon \]
Truncated and noisy Fourier transform

\[ y = TFx + \varepsilon \]

- \( x \in \mathbb{R}^N \): unknown image
- \( y, \varepsilon \in \mathbb{C}^M \): measurements, errors
- \( F \): Fourier matrix \((N \times N)\)
- \( T \): truncation matrix \((M \times N)\), e.g., \( T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \)

Difficulties: \( M \ll N \), noise
Different formulations

- Fourier synthesis (original formulation)
  \[ y = TFx + \varepsilon \]

- Interpolation – extrapolation: change of variable \( \tilde{x} = Fx \)
  \[ y = T\tilde{x} + \varepsilon \]

- Deconvolution: transformation of data
  - \( \tilde{y} = F^\dagger T^t y \)
  - \( H = F^\dagger T^t TF \)
  \[ \tilde{y} = Hx + \tilde{\varepsilon} \]

- A few simple properties
  - \( F^\dagger F = FF^\dagger = I \): orthonormality
  - \( T^t \): zero-padding matrix, \( M \leadsto N \) (\( T^t \) extends)
  - \( T^tT \): (diagonal) projection matrix, \( N \leadsto N \) (\( T^tT \) nullifies)
  - \( TT^t = I_M \)
Interferometry: illustration

True map ES

True map PS

Dirty beam

Dirty map ES

Dirty map PS

Dirty map PS + ES
Data based inversion and ill-posed character

- Deficient rank, missing data
  - $FT, T, H$: 1 singular value order $M$ and 0 order $N - M$

- Infinity of Least-Squares solution
  - $J_{LS}(x) = \| y - TFx \|^2$

- Other solutions: minimum norm solution, TSVD, (quasi) Wiener . . .
  - $J_{LS}(\hat{y}) = 0$
  - $J_{LS}\left(\hat{y} + [u - F^\dagger T^tTFu]\right) = 0$, for all map $u$

- Necessity of other information
Taking constraints into account: positivity and support

- **Notation**
  - $\mathcal{M}$: index set of the image pixels
  - $S, D \subseteq \mathcal{M}$: index set of a part of the image pixels

- **Investigated constraints here**
  - **Positivity**
    $$C_p : \forall p \in \mathcal{M}, \quad x_p \geq 0$$
  - **Support**
    $$C_s : \forall p \in \bar{S}, \quad x_p = 0$$

- **Extensions (non investigated here)**
  - **Template**
    $$\forall p \in \mathcal{M}, \quad t_p^- \leq x_p \leq t_p^+$$
  - **Partially known map**
    $$\forall p \in D, \quad x_p = m_p$$
Point sources + extended source

- **Double-model** [Ciuciu02, Samson03] et [Magain98, Pirzkal00]
  - \( x = x_e + x_p \)
  - Direct model \( y = TF(x_e + x_p) + \varepsilon \)
  - New indeterminations

- **Appropriate regularisation**
  - \( P_e(x_e) = \sum_{p \sim q} [x_e(p) - x_e(q)]^2 \)
  - \( P_p(x_p) = \)
Point sources + extended source

- **Double-model** [Ciuciu02, Samson03] et [Magain98, Pirzkal00]
  - $x = x_e + x_p$
  - Direct model $y = TF(x_e + x_p) + \epsilon$
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- **Appropriate regularisation**
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Double-model [Ciuciu02, Samson03] et [Magain98, Pirzkal00]

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Appropriate regularisation

- \( P_e(x_e) = \sum_{p \sim q} [x_e(p) - x_e(q)]^2 \)
- \( P_p(x_p) = \sum |x_p(n)| = \sum x_p(n) \)
Frequential analysis

Reduced frequency
Regularized criterion y regularized solution

Criterion: penalized, quadratic, strictly convex

\[ J(x_e, x_p) = \|y - T_F(x_e + x_p)\|^2 + \lambda_e \sum_{p \sim q} [x_e(p) - x_e(q)]^2 + \lambda_p \sum x_p(n) + \varepsilon_e \sum x_e(n)^2 + \varepsilon_p \sum x_p(n)^2 \]

Solution: unique constrained minimizer \( x = [x_e; x_p] \)

\[
(\hat{x}_e, \hat{x}_p) = \begin{cases} 
\arg \min J(x_e, x_p) \\
\text{s.t. } (C')
\end{cases}
= \begin{cases} 
\arg \min \frac{1}{2} x^T Q x + q^T x \\
\text{s.t. } \begin{cases} 
x_p = 0 & \text{for } p \in \bar{S} \\
x_p \geq 0 & \text{for } p \in M
\end{cases}
\end{cases}
\]
Positivity: one variable

- One variable: \( \alpha(t - \bar{t})^2 + \gamma \)

Non-constrained solution: \( \hat{t} = \bar{t} \)

Constrained solution: \( \hat{t} = \max[0, \bar{t}] \)

Active and inactive constraints
Two variables: 
\[ \alpha_1 (t_1 - \bar{t}_1)^2 + \alpha_2 (t_2 - \bar{t}_2)^2 + \beta (t_2 - t_1)^2 + \gamma \]

Sometimes / often difficult to deduce the constrained minimiser from the non-constrained one
Positivity: two variables (2)

- Two variables: \( \alpha_1(t_1 - \bar{t}_1)^2 + \alpha_2(t_2 - \bar{t}_2)^2 + \beta(t_2 - t_1)^2 + \gamma \)

- Constrained solution = Non-constrained solution (1)
- Constrained solution \( \neq \) Non-constrained solution (2)
  
  ... so active constraints
Two variables: \( \alpha_1(t_1 - \bar{t}_1)^2 + \alpha_2(t_2 - \bar{t}_2)^2 + \beta(t_2 - t_1)^2 + \gamma \)

Constrained solution \( \neq \) Non-constrained solution (2)  
... so active constraints

- Constrained solution \( \neq \) Projected non-constrained solution (2a)
  \[ (\hat{t}_1; \hat{t}_2) \neq (\max [0, \bar{t}_1]; \max [0, \bar{t}_2]) \]

- Constrained solution = Projected non-constrained solution (2b)
  \[ (\hat{t}_1; \hat{t}_2) = (\max [0, \bar{t}_1]; \max [0, \bar{t}_2]) \]
Numerical optimisation: state of the art

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<td>- Quadratic optimisation with linear constraints</td>
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<th>Difficulties</th>
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<tr>
<td>- $N \sim 1,000,000$</td>
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<td>- Constraints $\oplus$ non-separable variables</td>
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<th>Existing algorithms</th>
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<td>- Existing tools <em>with guaranteed convergence</em></td>
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<td>- [Bertsekas 95,99; Nocedal 00,08; Boyd 04,11]</td>
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<td>- Gradient projection methods, constrained gradient method</td>
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<td>- Broyden-Fletcher-Goldfarb-Shanno (BFGS) and limited memory</td>
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<td>- Interior points and barrier</td>
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<td>- Pixel-wise descent</td>
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<td>- <strong>Augmented Lagrangian, ADMM</strong></td>
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<td>- Constrained but separated $+$ non-separated but non-constrained</td>
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<td>- Partial solutions still through FFT</td>
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Lagrangians und penalisation

- Equality constraint: \( x_p = 0 \)

\[- \sum_{p \in \bar{S}} \ell_p x_p + \frac{1}{2} c \sum_{p \in \bar{S}} x_p^2 \]

- Inequality constraint: \((x_p \geq 0) \Leftrightarrow (s_p - x_p = 0 ; s_p \geq 0)\)

\[- \sum_{p \in S} \ell_p (x_p - s_p) + \frac{1}{2} c \sum_{p \in S} (x_p - s_p)^2 \]

- Globally

\[\mathcal{L}(x, s, \ell) = \frac{1}{2} x^t Q x + q^t x - \ell^t (x - s) + \frac{1}{2} c (x - s)^t (x - s)\]
Iterative algorithm

\[ \mathcal{L}(\bm{x}, s, \ell) = \frac{1}{2} \bm{x}^t \bm{Q} \bm{x} + \bm{q}^t \bm{x} - \ell^t (\bm{x} - s) + \frac{1}{2} \bm{c} (\bm{x} - s)^t (\bm{x} - s) \]

- Iterate three steps
  1. Unconstrained minimization of \( \mathcal{L} \) w.r.t. \( \bm{x} \)
     \[ \tilde{\bm{x}} = - (\bm{Q} + \bm{cI})^{-1} (\bm{q} + [\ell + \bm{c}s]) \quad (\equiv \text{FFT}) \]
  2. Minimization of \( \mathcal{L} \) w.r.t. \( s \), s.t. \( s_p \geq 0 \),
     \[ \tilde{s}_p = \begin{cases} \max (0, cx_p - \ell_p)/c & \text{for } p \in S \\ 0 & \text{for } p \in \bar{S} \end{cases} \]
  3. Update \( \ell \)
     \[ \tilde{\ell}_p = \begin{cases} \max (0, \ell_p - cx_p) & \text{for } p \in S \\ \ell_p - cx_p & \text{for } p \in \bar{S} \end{cases} \]
Details about $Q$ and $q$

- $q \rightsquigarrow$ dirty map:

\[
q = \left. \frac{\partial J}{\partial x} \right|_{x_e, x_p=0} = \begin{bmatrix}
\frac{\partial J}{\partial x_e} \\
\frac{\partial J}{\partial x_p}
\end{bmatrix} = -2 \begin{bmatrix}
\overset{\circ}{y} \\
\overset{\circ}{y} + \lambda_c \frac{1}{2}
\end{bmatrix}
\]

- $Q \rightsquigarrow$ dirty beam:

\[
Q = \frac{\partial^2 J}{\partial x^2} = \begin{bmatrix}
\frac{\partial^2 J}{\partial x_e^2} & \frac{\partial^2 J}{\partial x_e \partial x_p} \\
\frac{\partial^2 J}{\partial x_p \partial x_e} & \frac{\partial^2 J}{\partial x_p^2}
\end{bmatrix} = \begin{bmatrix}
H + \lambda_s D^t D & H \\
H & H + \varepsilon_s I
\end{bmatrix}
\]
Simulated data results

True extended object $x_e^*$

Estimated extended object $\hat{x}_e$

Dirty map

True point object $x_p^*$

Estimated point object $\hat{x}_p$
Simulated data results

True extended object $x^*_e$

Estimated extended object $\hat{x}_e$

True point object $x^*_p$

Estimated point object $\hat{x}_p$
Real data: first results

Dirty map

Estimated extended object $\hat{x}_e$

Estimated point object $\hat{x}_p$
Real data: first results

Dirty map

Estimated extended object $\hat{\mathbf{x}}_e$

Estimated point object $\hat{\mathbf{x}}_p$
Conclusions

Synthesis

- Direct model and inverse problem
  - Interpolation-extrapolation/deconvolution/Fourier synthesis
  - Double-model and appropriate regularisation: point/background
  - Positivity and support
- Optimisation: lagrangians
- Simulations et real data: interferometry in radio-astronomy

Perspectives

- Quantitative assessment
- Case of a single map: an extended source / a set of point sources
- Non quadratic penalty $\leadsto$ background resolution enhancement
- Data and/or sources “out grid”
- Hyperparameter estimation