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Imaging and inversion

Exercises

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Exercice I — *Circulant Matrices.* This exercise is devoted to studying the properties of circulant matrices, more specifically the ones that allow them to be easily diagonalised. This is an important aspect with a great impact on the practical implementation of various processing methods, the likes of which one finds in spectral analysis or signal and image deconvolution.

We consider the matrices to be of real elements and of size $N \times N$. First, we will look at circular shift matrices, and afterwards we will look at the circulant matrices.

Notice — The aim of this exercise is to practice your skills in manipulating matrices and vectors, to remind you of certain results in linear algebra one should already be acquainted with,. . . and to introduce some properties of circulant matrices that are useful for the rest of the course. It is not strictly speaking a "math exercice" devoted to derive or prove the mentioned properties and results.

— Circular Shift Matrices —

We shall start by analysing the circular shift matrix $\boldsymbol{P}$, defined by the general element p_{nm} (n is the row index and m is the column index):

$$p_{nm} = \begin{cases} p_{nm} = 1 & \text{if } m = n + 1, \ \text{ for } n = 1, 2, \ldots, N - 1 \\ p_{nm} = 1 & \text{if } n = N \ \text{ and } m = 1 \\ p_{nm} = 0 & \text{otherwise} \end{cases}$$

We denote by $\lambda_1, \lambda_2, \ldots, \lambda_N$ the eigenvalues of $\boldsymbol{P}$ and we also denote $\boldsymbol{\Lambda} = \mathrm{diag}\big[\lambda_1, \lambda_2, \ldots, \lambda_N\big]$ the diagonal matrix containing the eigenvalues of $\boldsymbol{P}$ on its diagonal. We impose $\lambda_1 = 1$.

1. *A quick insight on several properties of $\boldsymbol{P}$ illustrated for the case $N = 4$.*

1a. *Write out $\boldsymbol{P}$.*

1b. *Write out the result of $\boldsymbol{P}\boldsymbol{u}$, where $\boldsymbol{u}$ is a vector belonging to $\mathbb{R}^N$.*

1c. *Write out the powers of $\boldsymbol{P}$. Explain why $\boldsymbol{P}^N = \boldsymbol{I}$.*

1d. *Prove that $\boldsymbol{P}^{\mathrm{t}}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{P}^{\mathrm{t}} = \boldsymbol{I}$ and also determine $\boldsymbol{P}^{-1}$.*

2. *Now, we focus on determining the eigenvalues of $\boldsymbol{P}$. Consider an eigenvector $\boldsymbol{f}$ together with its associated eigenvalue λ: $\boldsymbol{P}\boldsymbol{f} = \lambda\boldsymbol{f}$. By examining this relation on a per component basis, determine the eigenvalues of $\boldsymbol{P}$.*

3. *As opposed to the previous question, here we will focus on determining the eigenvectors of $\boldsymbol{P}$.*

3a. *Consider the eigenspace associated to the eigenvalues $\lambda_1 = 1$, i.e. the space of vectors which are circular shift invariant. We denote by $\boldsymbol{f}_1$ the associated unitary eigenvector.*

3b. *Furthermore, for an eigenvalue λ_n and its associated eigenvector $\boldsymbol{f}_n$, determine the individual components $f_n(2), f_n(3), \ldots, f_n(N)$ of the vector $\boldsymbol{f}_n$ as a function of λ_n and the first component $f_n(1)$. Determine thus, the set of unitary eigenvectors $\boldsymbol{f}_n$, for $n = 1, 2, \ldots, N$.*

4. *Matrix of eigenvector, change of basis.*

4a. *The matrix $\boldsymbol{F}$ is used to collect all eigenvectors. Give an explicit formula for the individual entries of the matrix $\boldsymbol{F}$ and write out the product $\boldsymbol{X} = \boldsymbol{F}\boldsymbol{x}$ for a vector $\boldsymbol{x} \in \mathbb{R}^N$.*

4b. *Prove that $\boldsymbol{F}$ is an orthonormal matrix: $\boldsymbol{F}^\dagger \boldsymbol{F} = \boldsymbol{F}\boldsymbol{F}^\dagger = \boldsymbol{I}$.*

5. *Find the diagonalisation formula (in a matrix form) between the matrices $\boldsymbol{P}$, $\boldsymbol{F}$ and $\boldsymbol{\Lambda}$.*

— Circulant Matrices —

A circulant matrix $\boldsymbol{C}$ of size $N \times N$ is defined starting from a set of N scalar values $c_1, c_2, \ldots, c_N$ as follows:

$$\boldsymbol{C} = \sum_{n=1}^{N} c_n \, \boldsymbol{P}^{n-1} \tag{1}$$

involving the successive powers of matrix $\boldsymbol{P}$.

6. *Write out the matrix $\boldsymbol{C}$ for the case $N = 4$.*

7. *Diagonalising the matrix $\boldsymbol{C}$.*

7a. *Starting from the answer at point 5, find the matrix relation which expresses the diagonalisation of the matrix $\boldsymbol{C}$.*

7b. *Specify which are the eigenvalues and also make a connection with the Fourier transformation of the sequence of coefficients c_n.*

8. *Show how can one compute the circular convolutions $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$ and $\boldsymbol{v} = \boldsymbol{C}^{-1}\boldsymbol{u}$ by exploiting the diagonal form of $\boldsymbol{C}$.*

In conclusion, for a circulant matrix one can use the discrete Fourier transform (DFT) to mark it diagonal as well as to compute its eigenvalues. By taking advantage of the FFT algorithm to compute the DFT, one can perform the aforementioned operations in a very efficient manner. These results allow us to replace the inversion of a full matrix having a circulant structure with the inversion of a diagonal matrix, with the added cost of computing several FFT, thus enabling one to have fast computations of certain deconvolution methods.

Exercice II — *Quadratic Functions.* This exercise is devoted to the study of certain properties of quadratic functions of several variables, to the derivation of their gradient and Hessian matrix, and to finding their minimum value together with the minimiser. This sort of calculations arise in the development of estimators as minimisers of quadratic criteria and in the development of numerical optimisation algorithms. The aforementioned developments make their way in a series of different data processing methods, for instance in signal and image deconvolution.

Notice — The aim of this exercise is to remind you of certain results one should already be acquainted with, to practice your skills of manipulating such functions,. . . It is not strictly speaking a “math exercice” devoted to derive or prove the above mentioned properties and results.

Let φ be a function, $\varphi : \mathbb{R}^N \to \mathbb{R}$, which takes as input a vector $\boldsymbol{x} \in \mathbb{R}^N$ and produces as output the following scalar quantity

$$\varphi(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{t}} \boldsymbol{Q} \boldsymbol{x} + \boldsymbol{q}^{\mathrm{t}} \boldsymbol{x} + q_0$$

where q_0 is a scalar, $\boldsymbol{q}$ is a vector of size N and $\boldsymbol{Q}$ is a square symmetric matrix of size $N \times N$ strictly positive-definite. By default, the vectors are column vectors.

1. *Let's start with two particular scenarios for the general form of the φ function*

1a. *In the first one, $\boldsymbol{q} = 0$. Specify, without actually doing any calculations, the minimiser and the minimum value of φ*

1b. *In the second one, $\boldsymbol{Q}$ is a diagonal matrix. We shall denote by ρ_n, $n = 1, \dots, N$, the N elements on the diagonal of $\boldsymbol{Q}$. Find the minimiser of φ. Analyse also the case when $\boldsymbol{Q}$ is proportional to the identity matrix: $\boldsymbol{Q} = \rho \boldsymbol{I}$.*

2. *We shall now focus our attention to determining the gradient and Hessian matrix of φ.*

2a. *Given a point $\boldsymbol{x}_0 \in \mathbb{R}^N$ and a displacement $\boldsymbol{\varepsilon} \in \mathbb{R}^N$, write the resulting function $\varphi(\boldsymbol{x}_0 + \boldsymbol{\varepsilon})$ such that three distinctive terms with respect to $\boldsymbol{\varepsilon}$ appear: a constant one, a linear one and a quadratic one.*

2b. *From the result of the previous sub-question, determine the gradient $\boldsymbol{g}(\boldsymbol{x}_0)$ and the Hessian matrix $\boldsymbol{H}(\boldsymbol{x}_0)$ of φ at the point $\boldsymbol{x}_0$. The former is a vector of size N while the latter is a matrix of size $N \times N$.*

3. *After finding out the gradient and Hessian matrix we shall now determine the minimiser of φ.*

3a. *Determine the minimiser $\bar{\boldsymbol{x}}$ of φ utilising the results from the previous question. Explicit the obtained minimiser for the two particular cases from question 1.*

3b. *Rewrite φ under a canonical form*

$$\varphi(\boldsymbol{x}) = (\boldsymbol{x} - \boldsymbol{x}_0)^{\mathrm{t}} \boldsymbol{M} (\boldsymbol{x} - \boldsymbol{x}_0) + m_0 \tag{2}$$

by identifying $\boldsymbol{x}_0$, $\boldsymbol{M}$ and m_0. One can use several approaches: trial and error, identification,...

3c. *Give a graphical representation for the cases $N = 1$ and $N = 2$.*

4. *Directional study. Being given a point $\boldsymbol{y}_0 \in \mathbb{R}^N$ and a non-null direction $\boldsymbol{\delta} \in \mathbb{R}^N$, we shall consider a displacement from $\boldsymbol{y}_0$ in the direction $\boldsymbol{\delta}$ of amplitude $\eta \in \mathbb{R}$. We can construct a new function ψ defined on $\mathbb{R}$ taking values in $\mathbb{R}$ defined by $\psi(\eta) = \varphi(\boldsymbol{y}_0 + \eta\boldsymbol{\delta})$.*

4a. *Perform a thorough analysis of ψ, paying special attention to its derivatives.*

4b. *Specify the minimiser $\bar{\eta}$. Under what circumstances do we have $\bar{\eta} = 0$? What about $\bar{\eta} > 0$? And $\bar{\eta} < 0$?*

4c. *We shall finish by analysing the case when the displacement is performed in the direction opposite to the gradient: $\boldsymbol{\delta} = -\boldsymbol{g}(\boldsymbol{y}_0)$. Determine the optimal step size $\bar{\eta}$. We denote $\bar{\boldsymbol{y}}_0 = \boldsymbol{y}_0 + \bar{\eta}\boldsymbol{\delta}$.*

- *Prove that the gradient at $\bar{\boldsymbol{y}}_0$ and at $\boldsymbol{y}_0$ are orthogonal.*
- *Analyse what happens to $\bar{\boldsymbol{y}}_0$ in the special case when $\boldsymbol{Q}$ is proportional to the identity matrix $\boldsymbol{Q} = \rho\boldsymbol{I}$.*

Exercice III — *Signal Interpolation.*

> **Matrix first order Taylor expansion** — Let $\boldsymbol{A}$ be a square matrix of size $N \times N$ and $\boldsymbol{I}_N$ the identity matrix of size N, then the following is assumed to be true
>
> $$(\boldsymbol{I}_N + \varepsilon \boldsymbol{A})^{-1} \underset{\varepsilon=0}{\sim} \boldsymbol{I}_N - \varepsilon \boldsymbol{A}\,,$$
>
> which generalises the classical scalar result $(1+\varepsilon)^{-1} \underset{\varepsilon=0}{\sim} 1 - \varepsilon$.

We shall analyse a physical phenomenon described by a signal $x(t)$: temperature, pressure, voltage, current,... For this purpose we sample the signal with the sampling period T_s: $x_n = x(nT_\mathrm{s})$, for $n = 0, \ldots, N-1$ where N is an even number. Unfortunately, the measurement system at hand is not able to perfectly perform the required sampling, as its sampling period is less than T_s, meaning that it is able to measure one sample out of two. Apart from the previous problem, the measured samples are also affected by noise. Thus, the measurement system will provide us only with M noise corrupted samples y_m modelled by the following equation:

$$y_m = x_{2\,m} + e_m \quad \text{for } m = 0, 1, \ldots, M-1 \ \text{ and } N = 2M\,, \tag{3}$$

where the terms e_m represent the measurement errors or noise. In what follows, we shall model the errors as realisations of zero-mean independent random variables, all having the same variance r_e.

The underlying physical phenomenon is assumed to be slowly varying with respect to the sampling period: x_n varies slowly whereas e_n can vary fast.

— Rewriting —

We shall group the values of the signal x_n, the measurements y_m and the error components e_m into three vectors $\boldsymbol{x} = [x_0, \ldots, x_{N-1}]^\mathrm{t}$, $\boldsymbol{y} = [y_0, \ldots, y_{M-1}]^\mathrm{t}$ and $\boldsymbol{e} = [e_0, \ldots, e_{M-1}]^\mathrm{t}$. The problem we have to solve is to determine the components of the signal $\boldsymbol{x}$, all of them, starting from the available measurements $\boldsymbol{y}$ and without knowing the individual realisations of the error $\boldsymbol{e}$.

By rewriting equation (3) for $m = 0, \ldots, M-1$ using the vector notations previously introduced, one arrives at having to solve a problem of the form $\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{e}$, where the matrix $\boldsymbol{H}$ has a specific structure. For example, for the case $N = 6$ and $M = 3$:

$$\boldsymbol{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

1. *Calculate $\boldsymbol{H}^\mathrm{t}\boldsymbol{H}$ and $\boldsymbol{H}\boldsymbol{H}^\mathrm{t}$, for the case $N = 6, M = 3$ and in the general case. Are they inversible? Calculate $\boldsymbol{H}^\mathrm{t}\boldsymbol{y}$ and comment on its form.*

— Empirical Estimation —

2. *Propose one, or two, empirical estimator(s) $\widehat{\boldsymbol{x}}_\mathrm{E}$ for $\boldsymbol{x}$. One can choose to work in the time domain or in the frequency domain (or both).*

— Least Squares Estimation Method —

We shall first try to solve the posed problem using the least squares method. We have thus the following criterion:

$$\mathcal{J}_{\text{LS}}(\boldsymbol{x}) = \sum_{m=0}^{M-1} (y_m - x_{2m})^2 = (\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x})^{\text{t}}(\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2$$

and we shall define the least squares estimator as the minimiser of the above criterion, *i.e.* the value of $\boldsymbol{x}$ which minimises the above criterion.

3. *Show that the above criterion does not have a unique minimiser. Give the set of minimisers of it and provide appropriate remarks.*

— Penalised Least Squares Estimation Method I —

We shall look now at estimating $\boldsymbol{x}$ through the use of a regularised least squares approach. We modify the previously defined criterion by introducing a penalisation term on the quadratic norm of $\boldsymbol{x}$ (squared Euclidean norm)

$$\mathcal{R}_0(\boldsymbol{x}) = \sum_{n=0}^{N} x_n^2 = \boldsymbol{x}^{\text{t}}\boldsymbol{x} = \|\boldsymbol{x}\|^2 \ ,$$

thus obtaining the following penalised least squares criterion:

$$\mathcal{J}^0_{\text{PLS}}(\boldsymbol{x}) = \mathcal{J}_{\text{LS}}(\boldsymbol{x}) + \mu_0 \mathcal{R}_0(\boldsymbol{x})$$

and we define the penalised least squares estimator as its minimiser:

$$\widehat{\boldsymbol{x}}_0 = \arg\min_{\boldsymbol{x}} \mathcal{J}^0_{\text{PLS}}(\boldsymbol{x}) \, .$$

The regularisation parameter μ_0 (positive) allows us to tune the criterion.

4. *Discuss the choice of $\mathcal{J}^0_{\text{PLS}}$. What is the role of μ_0?*

5. *Calculate $\widehat{\boldsymbol{x}}_0$. Give an explicit writing of it by separating the even components from the odd ones. Analyse what happens when $\mu_0 = 0$ and when $\mu_0 \to \infty$.*

6. *Calculate the bias and the variance of the components of $\widehat{\boldsymbol{x}}_0$, separating again the even and odd components. Comment on the specificity of the obtained results. Analyse once more the results with respect to the two particular cases: $\mu_0 = 0$ and $\mu_0 \to \infty$.*

— Penalised Least Squares Estimation Method II —

We shall look now at the results based on a different penalty. The building blocks of this new term are the differences between the successive samples of $\boldsymbol{x}$:

$$\mathcal{R}_1(\boldsymbol{x}) = \sum_{n=1}^{N-1} (x_n - x_{n-1})^2 = \boldsymbol{x}^{\text{t}}\boldsymbol{D}^{\text{t}}\boldsymbol{D}\boldsymbol{x} = \|\boldsymbol{D}\boldsymbol{x}\|^2 \ .$$

7. *Write out the matrix $\boldsymbol{D}$. State some of its properties and particularities. Calculate $\boldsymbol{D}^{\text{t}}\boldsymbol{D}$.*

Thus, we obtain the new penalised least squares criterion:

$$\mathcal{J}_{\text{PLS}}^{1}(\boldsymbol{x}) = \mathcal{J}_{\text{LS}}(\boldsymbol{x}) + \mu_1 \mathcal{R}_1(\boldsymbol{x})$$

and we define the new estimate as:

$$\widehat{\boldsymbol{x}}_1 = \arg\min_{\boldsymbol{x}} \mathcal{J}_{\text{PLS}}^{1}(\boldsymbol{x})\,.$$

8. *Provide appropriate remarks with respect to the choice of $\mathcal{J}_{\text{PLS}}^{1}$ and the estimate $\widehat{\boldsymbol{x}}_1$. What is the purpose of μ_1?*

9. *Calculate $\widehat{\boldsymbol{x}}_1$.*

10. *Analyse the estimator $\widehat{\boldsymbol{x}}_1$ for the case when $\mu_1 \to 0$ and when $\mu_1 \to +\infty$ and comment on the specific form in both cases.*

11. *Calculate the bias and the covariance matrix of $\widehat{\boldsymbol{x}}_1$ and provide appropriate remarks with respect to the obtained results.*

12. *Provide an illustration of the various solutions, for various values of the regularisation parameter.*

Exercice IV — *Fourier Synthesis.*

Preliminary remarks : complex matrices. If $\boldsymbol{M}$ is a matrix having complex elements, then by $\boldsymbol{M}^{\dagger}$ we denote its conjugate transpose : $\boldsymbol{M}^{\dagger} = (\boldsymbol{M}^{\text{t}})^{*} = (\boldsymbol{M}^{*})^{\text{t}}$. We say $\boldsymbol{M}$ is Hermitian symmetric if $\boldsymbol{M} = \boldsymbol{M}^{\dagger}$. If $\boldsymbol{M}$ has real elements : $\boldsymbol{M}^{\dagger} = \boldsymbol{M}^{\text{t}}$.

Preliminary results : gradient and Hessian. Let $\boldsymbol{M}$ be a Hermitian symmetric square matrix of size $N \times N$ and $\boldsymbol{m}$ a vector of size N. The mappings φ and ψ from $\mathbb{C}^N$ to $\mathbb{R}$, which to a vector $\boldsymbol{u} \in \mathbb{C}^N$ associate :

$$\varphi(\boldsymbol{u}) \;=\; \boldsymbol{u}^{\dagger}\boldsymbol{M}\boldsymbol{u} \qquad \text{and} \qquad \psi(\boldsymbol{u}) \;=\; \boldsymbol{m}^{\dagger}\boldsymbol{u} + \boldsymbol{u}^{\dagger}\boldsymbol{m}$$

are twice differentiable. Their respective gradient is:

$$\frac{\partial}{\partial \boldsymbol{u}}\,\varphi(\boldsymbol{u}) \;=\; 2\boldsymbol{M}\boldsymbol{u} \qquad \text{and} \qquad \frac{\partial}{\partial \boldsymbol{u}}\,\psi(\boldsymbol{u}) \;=\; 2\boldsymbol{m}$$

and their respective Hessian matrix is :

$$\frac{\partial^2}{\partial \boldsymbol{u}^2}\,\varphi(\boldsymbol{u}) \;=\; 2\boldsymbol{M} \qquad \text{and} \qquad \frac{\partial^2}{\partial \boldsymbol{u}^2}\,\psi(\boldsymbol{u}) \;=\; \boldsymbol{0}\,.$$

The MRI is a modern medical imaging technique which provides high resolution images. From the point of view of the data processing involved in the image reconstruction, the problem falls in the class of « *Fourier synthesis* » (FS) problems: the observed data set represents an incomplete (and noise corrupted) set of Fourier transform coefficients of the unknown object.

1. *Provide a short explanation of why the* MRI *image reconstruction problem is an ill-posed one.*

Generally speaking, the Fourier coefficients that are observed are those corresponding to low frequencies. The coefficients corresponding to high frequencies are usually not observed.

2. *How does this uneven observation of frequencies manifest itself in the reconstructed image ?*

In practice the problem of FS is posed on a 2D or 3D domain. However, we tackle it on a 1D domain. Mathematically, we can formulate the problem as follows

$$\boldsymbol{y} = \boldsymbol{T}\boldsymbol{F}\boldsymbol{x} + \boldsymbol{e}\,, \tag{4}$$

where

- $\boldsymbol{y} \in \mathbb{C}^M$ is the vector containing the M observed data, $\boldsymbol{x} \in \mathbb{C}^N$ is the vector containing the N unknowns and $\boldsymbol{e} \in \mathbb{C}^M$ is the vector containing the associated M measurement errors
- $\boldsymbol{F}$ is the normalised DFT (Discrete Fourier Transform) matrix : $\boldsymbol{F}^\dagger\boldsymbol{F} = \boldsymbol{F}\boldsymbol{F}^\dagger = \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix of size N. Furthermore, it is a symmetric matrix $\boldsymbol{F}^{\mathrm{t}} = \boldsymbol{F}$
- $\boldsymbol{T}$ is the truncation matrix : it extracts from $\boldsymbol{F}\boldsymbol{x}$ the Fourier coefficients that do get observed and discards the ones that do not. For example, if $N = 8$, $M = 3$ and the system only observes the first, the second and the fourth coefficient, we have :

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. *Preliminary remarks and results, for a general value of N and M.*

3a. *What is the expression of the matrix $\boldsymbol{T}$ if all the coefficients are observed ? How about when only one out of two coefficients are observed ?*

3b. *Give the matrices $\boldsymbol{T}^{\mathrm{t}}\boldsymbol{T}$, $\boldsymbol{T}\boldsymbol{T}^{\mathrm{t}}$ and the vector $\bar{\boldsymbol{y}} = \boldsymbol{T}^{\mathrm{t}}\boldsymbol{y}$.*

3c. *Let $\boldsymbol{x}_0 \in \mathbb{C}^N$ be some vector and let $\boldsymbol{x} = \boldsymbol{F}^\dagger(\boldsymbol{I} - \boldsymbol{T}^{\mathrm{t}}\boldsymbol{T})\boldsymbol{F}\boldsymbol{x}_0$. For this object $\boldsymbol{x}$, what is the expression of the vector $\boldsymbol{y}$ containing the observed data ? Discuss the result.*

4. *Fourier Synthesis, "inter-extra-polation" and convolution.*

4a. *With the aid of the change of variable $\mathring{\boldsymbol{x}} = \boldsymbol{F}\boldsymbol{x}$, show that the FS (4) problem can be cast as an "inter-extra-polation" problem. Give a concise argument supporting this claim.*

4b. *Let $\widetilde{\boldsymbol{x}} = \boldsymbol{F}^\dagger\bar{\boldsymbol{y}}$. Give the relationship between $\widetilde{\boldsymbol{x}}$ and $\boldsymbol{x}$. Deduce that the FS problem (4) can be cast as a deconvolution problem as well. Give an argument supporting this new claim.*

The remainder of this problem is devoted to the development of estimators $\widehat{\boldsymbol{x}}$ for the unknwon object $\boldsymbol{x}$.

— Empirical estimation —

5. *Propose an empirical estimator $\widehat{\boldsymbol{x}}_{\mathrm{E}}$ of $\boldsymbol{x}$.*

6. *Determine the expression of $\mathring{\widehat{\boldsymbol{x}}}_{\mathrm{E}} = \boldsymbol{F}\widehat{\boldsymbol{x}}_{\mathrm{E}}$ and discuss the characteristics of $\widehat{\boldsymbol{x}}_{\mathrm{E}}$ from a resolution point of view.*

— Least squares estimation method —

We seek to estimate $\boldsymbol{x}$ using the LS method. We introduce a criterion $\mathcal{J}_{\mathrm{LS}}$ obtained from equation (4) :

$$\mathcal{J}_{\mathrm{LS}}(\boldsymbol{x}) = (\boldsymbol{y} - \boldsymbol{T}\boldsymbol{F}\boldsymbol{x})^{\dagger}(\boldsymbol{y} - \boldsymbol{T}\boldsymbol{F}\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{T}\boldsymbol{F}\boldsymbol{x}\|^2 \ .$$

7. *Show that $\mathcal{J}_{LS}$ does not have a unique minimiser. What does this imply for the least squares estimation method ?*

8. *Give and discuss $\mathcal{J}_{LS}(\widetilde{\boldsymbol{x}})$.*

— Penalised least squares estimation method —

We seek now to estimate $\boldsymbol{x}$ using the PLS method. We introduce a penalisation term $\mathcal{P}$ taking into account the difference between successive samples of $\boldsymbol{x}$ (in a circular case, *i.e.*, $x_{N+1} = x_1$) :

$$\mathcal{P}(\boldsymbol{x}) = \sum_{n=1}^{N} |\, x_{n+1} - x_n \,|^2 = \boldsymbol{x}^{\dagger}\boldsymbol{D}^{\dagger}\boldsymbol{D}\boldsymbol{x} \, .$$

9. *Construct $\mathcal{J}_{PLS}$ the corresponding* PLS *criterion and let μ denote the regularisation parameter.*

10. *Write out the matrix $\boldsymbol{D}$. Highlight its particularities.*

11. *Specify the role and the influence of μ.*

12. *Give $\widehat{\boldsymbol{x}}_{PLS}$.*

13. *Determine $\mathring{\widehat{\boldsymbol{x}}}_{PLS} = \boldsymbol{F}\widehat{\boldsymbol{x}}_{PLS}$ and discuss the characteristics of $\widehat{\boldsymbol{x}}_{PLS}$ from a resolution point of view.*

14. *Analyse the case when μ tends to 0 and when μ tends to $+\infty$.*

15. *Propose a class of methods which enables to improve the resolution of the reconstructed images.*

Exercice V — *Signal Denoising: non-linear approach.* We shall analyse a physical phenomenon described by a signal $x(t)$: voltage, current, electromagnetic wave,. . . For this purpose, we are going to take N measurements y_n, for $n = 1, \ldots, N$. The measurements are taken at regular sampling intervals given by the sampling period T_{s} and are affected by errors (in an additive manner):

$$y_n = x_n + b_n = x(nT_{\mathrm{s}}) + b_n \, . \tag{5}$$

Overall, the underlying physical phenomenon is assumed to be slowly varying with respect to the sampling frequency: x_n evolves slowly whereas b_n can evolve quickly. From time to time though, the signal may exhibit sudden changes, discontinuities in intensity.

We shall collect all N samples of x_n, y_n and b_n in three vectors $\boldsymbol{x} = [x_1, \ldots, x_N]^{\mathrm{t}}$, $\boldsymbol{y} = [y_1, \ldots, y_N]^{\mathrm{t}}$ and $\boldsymbol{b} = [b_1, \ldots, b_N]^{\mathrm{t}}$. The problem we wish to solve is to estimate $\boldsymbol{x}$, which characterises the physical phenomenon under study, starting from the measurements $\boldsymbol{y}$ without actually knowing the error terms $\boldsymbol{b}$.

If necessary, in the following, we shall model the measurement errors as realisations of independent zero-mean random variables, each having the same variance r_{b}.

1a. *Write (5) for $n = 1, 2, \ldots, N$ in a vector form. The problem thus appears as a particular case of the more general linear problem $\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{b}$. Give the matrix $\boldsymbol{H}$ in this case.*

1b. *Provide appropriate remarks as to why the denoising problem is indeterminate.*

— The Method of Least Squares —

As a first solution to this problem we shall use the least squares method. It is defined by the following criterion:

$$Q_{\text{LS}}(\boldsymbol{x}) = \sum_{n=1}^{N} (x_n - y_n)^2 \,,$$

and the least squares estimator is defined as its minimiser: $\widehat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} Q_{\text{LS}}(\boldsymbol{x})$.

2. *Find the expression of the least squares estimator $\widehat{\boldsymbol{x}}$ and provide appropriate remarks with respect to it.*

— Quadratic penalisation —

As a second solution we shall look at estimating $\boldsymbol{x}$ by using a penalised least squares approach. As such we will introduce a penalisation on the quadratic norm (Euclidean norm) of the differences between successive samples of $\boldsymbol{x}$:

$$P_2(\boldsymbol{x}) = \sum_{n=1}^{N-1} (x_{n+1} - x_n)^2 \,.$$

We obtain thus the following penalised least squares criterion:

$$Q_2(\boldsymbol{x}) = Q_{\text{LS}}(\boldsymbol{x}) + \mu P_2(\boldsymbol{x}) \,,$$

and we define the penalised least squares estimator as its minimiser $\widehat{\boldsymbol{x}}_2 = \arg\min_{\boldsymbol{x}} Q_2(\boldsymbol{x})$. The extra positive valued parameter μ, known as the regularisation parameter, is a parameter that allows tuning of the method.

3. *Provide appropriate remarks with respect to the choice of Q_2. What is the influence of the extra parameter μ?*

4. *Find the minimiser $\widehat{\boldsymbol{x}}_2$ and provide appropriate remarks with respect to it. Enumerate several algorithms to actually compute it.*

5. *Particular cases: what happens when $\mu \sim 0$ and when $\mu \sim \infty$?*

— Non-Quadratic penalisation —

To better preserve discontinuities that are likely to be present in the signal $\boldsymbol{x}$, we shall modify the imposed penalisation by introducing three distinct potential functions:

$$\varphi_1(\delta) = |\,\delta\,| \quad , \quad \varphi_{2/1}(\delta) = \begin{cases} \delta^2 & \text{if } |\,\delta\,| \leqslant s \\ 2s\,|\,\delta\,| - s^2 & \text{if } |\,\delta\,| \geqslant s \end{cases} \quad \text{and} \quad \varphi_{2/0}(\delta) = \begin{cases} \delta^2 & \text{if } |\,\delta\,| \leqslant s \\ s^2 & \text{if } |\,\delta\,| \geqslant s \end{cases}$$

and for notational convenience, we state $\varphi_2(\delta) = \delta^2$.

6a. *Plot the four potential functions. Determine the first and second order derivatives and plot them. Identify the points of discontinuity and/or non-differantiability (first and second order).*

6b. *For $\varphi_{2/1}$ and $\varphi_{2/0}$ provide the resulting potential functions in the case $s \sim \infty$. Also, for $\varphi_{2/1}$ provide the resulting potential functions for the case $s \sim 0$.*

6c. *From the four potential function, identify the ones that are convex and the ones that aren't. Make also a distinction among the convex ones whether or not their are convex in the strict sense.*

For each of the above defined potential functions we shall construct a penalisation term:

$$P_\bullet(\boldsymbol{x}) = \sum_{n=1}^{N-1} \varphi_\bullet\left(x_{n+1} - x_n\right) ,$$

for $\bullet = 1$, $\bullet = 2/1$ and $\bullet = 2/0$ together with the corresponding penalised criterion:

$$Q_\bullet(\boldsymbol{x}) = Q_{\text{LS}}(\boldsymbol{x}) + \mu P_\bullet(\boldsymbol{x}) .$$

The estimated signal $\widehat{\boldsymbol{x}}_\bullet$ is the one that minimises: $\widehat{\boldsymbol{x}}_\bullet = \arg\min_{\boldsymbol{x}} Q_\bullet(\boldsymbol{x})$.

7. *Predict the behaviour of the three estimators for the case when $\mu \sim 0$ and when $\mu \sim \infty$. Further predict the behaviour of $\widehat{\boldsymbol{x}}_{2/1}$ and $\widehat{\boldsymbol{x}}_{2/0}$ as $s \sim \infty$.*

8. *This question will focus on the analysis of the influence of the individual components x_{n_0} on $Q_\bullet$ (with $n_0 \neq 1$ and $n_0 \neq N$). To this end, let us introduce*

$$q_\bullet(x|x_{\text{left}}, x_{\text{right}}, y) = (x-y)^2 + \mu\left[\varphi_\bullet\left(x - x_{\text{left}}\right) + \varphi_\bullet\left(x - x_{\text{right}}\right)\right]$$

which whenever not ambiguous, shall be simply referred to as $q_\bullet(x)$.

8a. *Prove that $q_\bullet(x_{n_0})$ equals $Q_\bullet(\boldsymbol{x})$ up to a certain additive constant (that does not depend on x_{n_0}). Identify the terms x_{left} and x_{right}.*

8b. *Analyse each of the resulting four functions $q_\bullet(x)$. One can restrict the analysis just for the case $x_{\text{left}} \leqslant x_{\text{right}}$.*

8c. *Find the minimiser $\overline{x}_\bullet$ of $q_\bullet$. One can start with the case $\bullet = 2$, then move to the case $\bullet = 1$ and then move to the remaining two cases.*

8d. *Deduce qualitative properties one would expect for each of the estimated signal regarding the preservation of discontinuities.*

9. *Deduce a minimisation algorithm for $Q_\bullet$. In which case does it provide us with a global minimiser?*

10. *Try to propose a method to estimate the hyper-parameters μ and s while also stating what difficulties you think one would encounter trying to implement them.*

11. *Propose other denoising methods and state for each, with respect to the properties of the signal to be denoised, under what circumstances it is best to be used.*

Exercice VI — *Legendre transform of quadratic functions.* The aim of this exercise is to practice your skill of manipulating the Legendre Transform (LT). To this end, we will deal with a simple scalar quadratic function. This is an important aspect with a great impact on the practical implementation of various processing methods, for instance in signal and image deconvolution.

Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_+^\star$ and $x_0 \in \mathbb{R}$. Let us consider the quadratic function f from $\mathbb{R}$ onto $\mathbb{R}$ defined by

$$f(x) = \alpha + \frac{1}{2}\beta(x - x_0)^2 \,,$$

for $x \in \mathbb{R}$.

1. *For a start, plot the function $f(x)$ as well as its first and second derivatives. Then perform a quick and proper analysis of the influence of α, β and x_0. Discuss the particular cases where $\alpha = 0$, $\beta = 1$ and $x_0 = 0$.*

We shall now recall the definition of $f^\star$, the LT of f. Itself is a function from $\mathbb{R}$ onto $\mathbb{R}$ and its expression is given by

$$f^\star(t) \;=\; \sup_{x\in\mathbb{R}} \left[\, xt - f(x) \,\right]$$

that is, it is defined as the result of a function maximisation problem. Let us denote $g_t(x) = xt - f(x)$.

2a. *Find $f^\star$. Also give its first and second derivatives.*

2b. *Analyse the influence of α, β et x_0. Discuss the special cases where $\alpha = 0$, $\beta = 1$ and $x_0 = 0$.*

Exercice VII — *Legendre transform of transforms.* Let f be a strictly convex function defined on $\mathbb{R}$ and taking values in $\mathbb{R}$ and consider the following four real numbers $\alpha \in \mathbb{R}$, $\beta, \gamma \in \mathbb{R}_+^\star$ and $x_0 \in \mathbb{R}$. We define the following three new functions:

$$\begin{aligned} u(x) &= \alpha + \beta\, f(x) \\ v(x) &= f(x - x_0) \\ w(x) &= f(\gamma x) \end{aligned}$$

by means of scaling and/or shifting (vertically or horizontally) the function f.

We remind one that the Lengendre transform $f^\star$ of f is a function defined on $\mathbb{R}$ and taking values in $\mathbb{R}$ which is obtained as the result of a maximisation problem. For $t \in \mathbb{R}$ we have:

$$f^\star(t) \;=\; \sup_{x\in\mathbb{R}} \left[\, xt - f(x) \,\right] .$$

1. *Determine the Lengendre transforms $u^\star$, $v^\star$ and $w^\star$ as functions of $f^\star$.*

Exercice VIII — *Legendre transform for absolute penalty.* The aim of this exercise is to calculate the Legendre Transform (LT) of the simple scalar absolute value function defined by: $f(x) = |x|$ for $x \in \mathbb{R}$.

1. *For a start, plot the function $f(x)$. Is it a convex function ? Large sense or strict sense convex ?*

We shall now recall an extended definition of $f^\star$, the LT of f. Itself is defined as a function from $\mathbb{R}$ onto $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ and its expression is given by

$$f^\star(t) \;=\; \sup_{x\in\mathbb{R}} \left[\, xt - f(x) \,\right]$$

that is, it is defined as the result of a function maximisation problem. Let us denote $g_t(x) = xt - f(x)$.

2. *Prove that* $f^\star$ *is* 0 *for* $t \in [-1, +1]$ *and* $+\infty$ *if not.*

Exercice IX — *Quadratic criterion and linear constraints.* The purpose of this exercise is to familiarise oneself with the concept of constrained optimisation. It arises in many applications (instrument development, command and control, distribution over a network, finance, production planning and control,...) which are crucial in a vast array of domains such as telecommunications, energy and transport, health...It is also a building block in many data processing methods, specifically in the construction of estimators as minimisers of criteria involving constraints.

The literature abounds [1, 2, 3] in numerical constrained optimisation tools including those based on projected gradient or on constraint gradient, introduction of barrier functions, the methods based on Lagrange multipliers. We will focus on the latter.

Preliminary result: constrained optimisation and Lagrange multipliers. Let us consider a function F of a variable $\boldsymbol{x} \in \mathbb{R}^P$ with real values. Let us also consider Q equations $C_q(\boldsymbol{x}) = 0$, for $q = 1, \ldots, Q$. We are interested in finding $\boldsymbol{x}_\mathrm{o}$, the minimizer of F with respect to $\boldsymbol{x}$ subject to the Q contraints $C_q(\boldsymbol{x}) = 0$, that is to say the solution for the problem $(\mathcal{P})$

$$(\mathcal{P}) \; : \quad \boldsymbol{x}_\mathrm{o} = \arg\min_{\boldsymbol{x}\in\mathbb{R}^P} \begin{cases} F(\boldsymbol{x}) \\ \text{s.t. } \; C_q(\boldsymbol{x}) = 0, \;\; \text{for } q = 1, \ldots, Q \end{cases}$$

where s.t. stands for *subject to*.
Let us introduce Q parameters collected in a vector $\boldsymbol{\ell} = [\ell_1, \ldots, \ell_Q]$, referred to as Lagrange multipliers and build an extended function referred to as the Lagrangian:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\ell}) = F(\boldsymbol{x}) + \sum_{q=1}^{Q} \ell_q \, C_q(\boldsymbol{x})$$

that is a function of $\boldsymbol{x}$ and $\boldsymbol{\ell}$. Let us minimize $\mathcal{L}(\boldsymbol{x}, \boldsymbol{\ell})$ with respect to $\boldsymbol{x}$ (for a fixed value of $\boldsymbol{\ell}$) and denote $\boldsymbol{x}_\mathrm{m}(\boldsymbol{\ell})$ the minimizer (which is natürlich a function of $\boldsymbol{\ell}$):

$$\boldsymbol{x}_\mathrm{m}(\boldsymbol{\ell}) = \arg\min_{\boldsymbol{x}\in\mathbb{R}^P} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\ell}) \,. \tag{6}$$

Let us substitute this value of $\boldsymbol{x}$ in the Lagrangian: this defines a function of $\boldsymbol{\ell}$ exclusively

$$\widetilde{\mathcal{L}}(\boldsymbol{\ell}) = \mathcal{L}(\boldsymbol{x}_\mathrm{m}(\boldsymbol{\ell}), \boldsymbol{\ell})$$

called the *dual function*. In order to obtain the adequate value of $\boldsymbol{\ell}$ to ensure that the constraints are satisfied, maximisation of the dual function $\widetilde{\mathcal{L}}$ with respect to $\boldsymbol{\ell}$ is required:

$$\boldsymbol{\ell}_\mathrm{m} = \arg\max_{\boldsymbol{\ell}\in\mathbb{R}^Q} \widetilde{\mathcal{L}}(\boldsymbol{\ell}) \,.$$

Finally, the constrained minimizer is then obtained by substitution of $\boldsymbol{\ell}_\mathrm{m}$ in the optimum with respect to $\boldsymbol{x}$ given by (6):

$$\boldsymbol{x}_\mathrm{o} = \boldsymbol{x}_\mathrm{m}(\boldsymbol{\ell}_\mathrm{m}) \,.$$

This result is valid under some hypothesis regarding F et C_p that are met in this exercise.

Notice — The aim of this exercise is to remind you of certain results one should already be acquainted with, to practice your skills of manipulating linear constraints and quadratic functions. It is not strictly speaking a "math exercice" in itself. On could refer to textbooks such as [1, 2, 3] for deep analysis.

Let φ be a function, $\varphi : \mathbb{R}^N \to \mathbb{R}$, which takes as input a vector $\boldsymbol{x} \in \mathbb{R}^N$ and produces as output the following scalar quantity

$$\varphi(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^{\mathrm{t}}\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{q}^{\mathrm{t}}\boldsymbol{x} + q_0$$

where q_0 is a scalar, $\boldsymbol{q}$ is a vector of size N and $\boldsymbol{Q}$ is a square symmetric matrix of size $N \times N$ strictly positive-definite. Consider a matrix $\boldsymbol{A}$ of size $M \times N$ and a vector $\boldsymbol{a}$ of size M. We are interested in the set of $\boldsymbol{x} \in \mathbb{R}^N$ that satisfies

$$\boldsymbol{A}\boldsymbol{x} - \boldsymbol{a} = \boldsymbol{0}\,.$$

This relationship represents a vectorial constraint that collects M scalar constraints given by M linear combinations of the N components of $\boldsymbol{x}$. $\boldsymbol{A}$ is assumed to be full rank, *i.e.,* the rank of $\boldsymbol{A}$ is $\min(M, N)$. Usually, $M < N$. The exercise deals with the minimization of φ including the constraint $\boldsymbol{A}\boldsymbol{x} - \boldsymbol{a} = \boldsymbol{0}$, that is to say the problem

$$(\mathcal{P}) \;:\quad \boldsymbol{x}_{\mathrm{o}} = \underset{\boldsymbol{x}\in\mathbb{R}^N}{\arg\min} \begin{cases} \varphi(\boldsymbol{x}) \\ \text{s.t. } \boldsymbol{A}\boldsymbol{x} - \boldsymbol{a} = \boldsymbol{0} \end{cases}$$

and focuses on its solution obtained using the method of Lagrange mutipliers.

1. *Introductory examples and particular cases.*

1a. *Give the solution of the problem if there is no constraint.*

1b. *Let us study the degenerated case where $M = N$, $\boldsymbol{a} = \boldsymbol{0}_N$ is the null vector and $\boldsymbol{A} = \boldsymbol{I}_N$ is the identity matrix of size N. Give the solution of the problem.*

1c. *Que pasa si $M = N$ y si $\boldsymbol{A}$ es inversible ?*

1d. *In the two-dimensionnal case, $N = 2$ (two variables) and $M = 1$ (one constraint), give a graphical illustration of the problem and of its solution.*

2. *Write the Lagrangian $\mathcal{L}(\boldsymbol{x}, \boldsymbol{\ell})$. Is it a quadratic or a linear function with respect to $\boldsymbol{x}$? What about with respect to $\boldsymbol{\ell}$? And finally with respect to $(\boldsymbol{x}, \boldsymbol{\ell})$ jointly ?*

3. *Give the minimizer $\boldsymbol{x}_{\mathrm{m}}(\boldsymbol{\ell})$ of $\mathcal{L}(\boldsymbol{x}, \boldsymbol{\ell})$ w.r.t. $\boldsymbol{x}$. Note that it is a linear function of $\boldsymbol{\ell}$.*

4. *Build the dual function $\widetilde{\mathcal{L}}(\boldsymbol{\ell}) = \mathcal{L}(\boldsymbol{x}_{\mathrm{m}}(\boldsymbol{\ell}), \boldsymbol{\ell})$ by replacing $\boldsymbol{x}_{\mathrm{m}}(\boldsymbol{\ell})$ in the Lagrangian $\mathcal{L}(\boldsymbol{x}, \boldsymbol{\ell})$.*

5. *Give the maximizer $\boldsymbol{\ell}_{\mathrm{m}}$ of the dual function $\widetilde{\mathcal{L}}(\boldsymbol{\ell})$ w.r.t. $\boldsymbol{\ell}$.*

6. *Finally, deduce the constrained minimizer by replacing $\boldsymbol{\ell}_{\mathrm{m}}$ in the expression of $\boldsymbol{x}_{\mathrm{m}}(\boldsymbol{\ell})$. Study the special case $M = N$ and $\boldsymbol{A}$ invertible in relation with question 1c.*

7. *Optional questions.*

7a. *Do you think the constrained minimum is larger or smaller than the unconstrained one? Check your answer by calculating the two values of the function φ.*

7b. *In the special case $M = N$ and $\boldsymbol{A}$ invertible (in relation with question 1c and 6), check that the minimum is $\varphi(\boldsymbol{A}^{-1}\boldsymbol{a})$.*

Exercice X — *Quadratic criterion and linear constraint, a two-one dimensional case.* This exercise deals with a constrained optimisation problem. More specifically, a two dimensional quadratic criterion and a one dimensional linear constraint.

One the one hand, let φ be a function, $\varphi : \mathbb{R}^2 \to \mathbb{R}$, which takes as input a couple of real numbers (x, y) and produces as output the following real number

$$\varphi(x, y) = 1 + (x^2 + y^2)/2 \,.$$

On the other hand, let us consider the set of couples of real numbers (x, y) that satisfy the constraint

$$(\mathcal{C}) \;:\; y = s\,x + o$$

where $s \in \mathbb{R}$ is a slope and $o \in \mathbb{R}$ is an offset.

The exercise deals with the minimization of φ including the constraint $(\mathcal{C})$, that is to say the problem

$$(\mathcal{P}) \;:\; (x_\star, y_\star) = \operatorname*{arg\,min}_{x,y \in \mathbb{R}} \begin{cases} \varphi(x, y) \\ \text{s.t. } y = s\,x + o \end{cases}$$

and focuses on its solution based on the method of Lagrange mutipliers, in addition to a direct approach.

1. *Give the unconstrained minimizer and the unconstrained minimum. The rest of the exercise is devoted to the constrained case.*

2. *Give a graphical illustration of the problem and of its solution.*

3. *Propose a direct solution, by substitution of the constraint $(\mathcal{C})$ in φ.*

4. *Lagrange mutipliers approach.*

4a. *Write the Lagrangian $\mathcal{L}(x, y, \lambda)$. Is it a quadratic or a linear function with respect to (x, y) ? What about with respect to λ ?*

4b. *Give the minimizer $x_{\mathrm{m}}(\lambda), y_{\mathrm{m}}(\lambda)$ of $\mathcal{L}(x, y, \lambda)$ w.r.t. (x, y). Is it a quadratic or a linear function with respect to λ?*

4c. *Build the dual function $\widetilde{\mathcal{L}}(\lambda) = \mathcal{L}(x_{\mathrm{m}}(\lambda), y_{\mathrm{m}}(\lambda), \lambda)$ by replacing $(x_{\mathrm{m}}(\lambda), y_{\mathrm{m}}(\lambda))$ in the Lagrangian $\mathcal{L}(x, y, \lambda)$.*

4d. *Give the maximizer λ_{m} of the dual function $\widetilde{\mathcal{L}}(\lambda)$ w.r.t. λ.*

4e. *Finally, deduce the constrained minimizer $(x_\star, y_\star)$ by replacing λ_{m} in the expression of $(x_{\mathrm{m}}(\lambda), y_{\mathrm{m}}(\lambda))$.*

5. *Do you think the constrained minimum is larger or smaller than the unconstrained one? Check your answer by calculating the two values of the function φ.*

6. *Compare the (constrained) minimum of the primal function φ and the (unconstrained) maximum of the dual function $\widetilde{\mathcal{L}}(\lambda)$.*

7. *Comment the special cases $o = 0$ and $s = 0$.*

Exercice XI — *Quadratic criterion and linear constraint, a two-one dimensional case.* This exercise deals with a constrained optimisation problem. More specifically, a two dimensional quadratic criterion and a one dimensional linear constraint.

One the one hand, let φ be a function, $\varphi : \mathbb{R}^2 \to \mathbb{R}$, which takes as input a couple of real numbers (x, y) and produces as output the following real number

$$\varphi(x, y) = \frac{1}{2} \left[(x-1)^2 + y^2 \right] .$$

On the other hand, let us consider the set of couples of real numbers (x, y) that satisfy the constraint

$$(\mathcal{C}) \; : \; y = s\, x$$

where $s \in \mathbb{R}$ is a slope.

The exercise deals with the minimization of φ including the constraint $(\mathcal{C})$, that is to say the problem

$$(\mathcal{P}) \; : \quad (x_\star, y_\star) = \underset{x,y \in \mathbb{R}}{\arg\min} \begin{cases} \varphi(x, y) \\ \text{s.t. } \; y = s\, x \end{cases}$$

and focuses on its solution based on the method of Lagrange mutipliers, in addition to a direct approach.

1. *Give the unconstrained minimizer and the unconstrained minimum. The rest of the exercise is devoted to the constrained case.*

2. *Give a graphical illustration of the problem and of its solution.*

3. *Propose a direct solution, by substitution of the constraint $(\mathcal{C})$ in φ.*

4. *Lagrange mutipliers approach.*

4a. *Write the Lagrangian $\mathcal{L}(x, y, \lambda)$. Is it a quadratic or a linear function with respect to (x, y) ? What about with respect to λ ?*

4b. *Give the minimizer $x_\mathrm{m}(\lambda), y_\mathrm{m}(\lambda)$ of $\mathcal{L}(x, y, \lambda)$ w.r.t. (x, y). Is it a quadratic or a linear function with respect to λ?*

4c. *Build the dual function $\widetilde{\mathcal{L}}(\lambda) = \mathcal{L}(x_\mathrm{m}(\lambda), y_\mathrm{m}(\lambda), \lambda)$ by replacing $(x_\mathrm{m}(\lambda), y_\mathrm{m}(\lambda))$ in the Lagrangian $\mathcal{L}(x, y, \lambda)$.*

4d. *Give the maximizer λ_m of the dual function $\widetilde{\mathcal{L}}(\lambda)$ w.r.t. λ.*

4e. *Finally, deduce the constrained minimizer $(x_\star, y_\star)$ by replacing λ_m in the expression of $(x_\mathrm{m}(\lambda), y_\mathrm{m}(\lambda))$.*

5. *Do you think the constrained minimum is larger or smaller than the unconstrained one? Check your answer by calculating the two values of the function φ.*

6. *Compare the (constrained) minimum of the primal function φ and the (unconstrained) maximum of the dual function $\widetilde{\mathcal{L}}(\lambda)$.*

7. *Comment the special cases $s = 1$ and $s = 0$. And the case $s = +\infty$.*

References

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