Image restoration

— Concluding remarks and perspectives —

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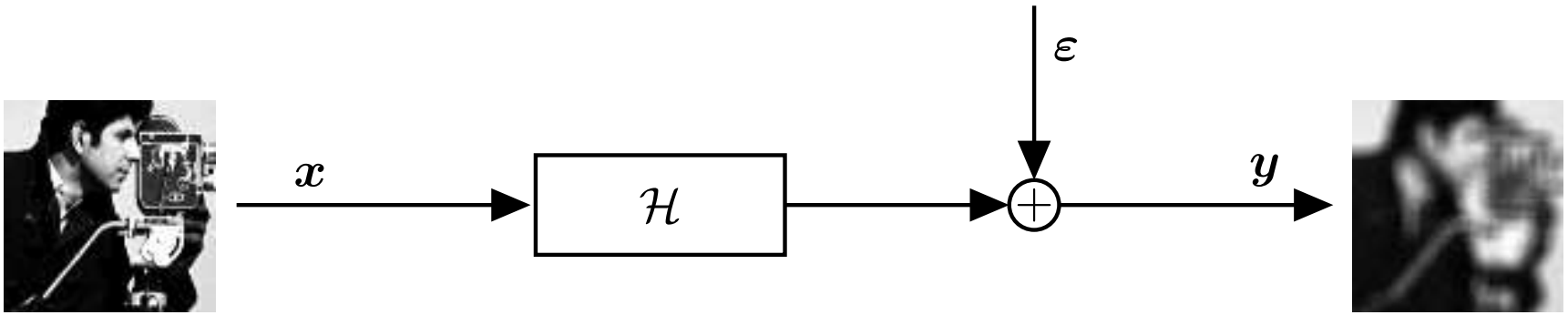
Convolution / Deconvolution

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{\varepsilon} = \boldsymbol{h} \star \boldsymbol{x} + \boldsymbol{\varepsilon}$$



$$\widehat{\boldsymbol{x}} = \widehat{\mathcal{X}}(\boldsymbol{y})$$

Restoration, deconvolution-denoising

- General problem: ill-posed inverse problems, *i.e., lack of information*
- Methodology: regularisation, *i.e., information compensation*
  - Specificity of the inversion / reconstruction / restoration methods
  - Trade off and tuning parameters
- Limited quality results

Three solutions: various penalties and constraints

- Smoothness and quadratic penalty

$$\mathcal{J}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \sum (x_p - x_q)^2 = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \, \|\boldsymbol{D}\boldsymbol{x}\|^2$$

- Edge preservation and Huber penalty

$$\mathcal{J}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \sum \varphi(x_p - x_q)$$

- Constraints: positivity and support

$$\begin{cases} \mathcal{J}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \, \|\boldsymbol{D}\boldsymbol{x}\|^2 \\ \text{s.t.} \begin{cases} x_p = 0 & \text{for } p \in \bar{\mathcal{S}} \\ x_p \geqslant 0 & \text{for } p \in \mathcal{M} \end{cases} \end{cases}$$

Three solutions: efficient computations

- Smoothness and quadratic penalty

$$\mathcal{J}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \sum (x_p - x_q)^2 = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \, \|\boldsymbol{D}\boldsymbol{x}\|^2$$

- Edge preservation and Huber penalty

$$\mathcal{J}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \sum \varphi(x_p - x_q)$$

  - Extended criterion, Legendre transform and half-quadratic idea

$$\tilde{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{a}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \sum \frac{1}{2} \left[(x_p - x_q) - a_{pq}\right]^2 + \zeta(a_{pq})$$

- Constraints: positivity and support

$$\begin{cases} \mathcal{J}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \, \|\boldsymbol{D}\boldsymbol{x}\|^2 \\ \text{s.t.} \begin{cases} x_p = 0 & \text{for } p \in \bar{\mathcal{S}} \\ x_p \geqslant 0 & \text{for } p \in \mathcal{M} \end{cases} \end{cases}$$

  - Extended criterion, augmented Lagrangian and ADMM

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{\ell}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \, \|\boldsymbol{D}\boldsymbol{x}\|^2 + \rho \, \|\boldsymbol{x} - \boldsymbol{a}\|^2 + \boldsymbol{\ell}^{\mathrm{t}}(\boldsymbol{x} - \boldsymbol{a})$$

Quadratic penalty: fast computation by FFT

- Reminder: least squares and quadratic penalty:

$$\mathcal{J}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \, \|\boldsymbol{D}\boldsymbol{x}\|^2$$

- Restored image

$$\begin{aligned} \widehat{\boldsymbol{x}} &= \arg\min_{\boldsymbol{x}} \mathcal{J}(\boldsymbol{x}) \\ (\boldsymbol{H}^{\mathrm{t}}\boldsymbol{H} + \mu \boldsymbol{D}^{\mathrm{t}}\boldsymbol{D}) \, \widehat{\boldsymbol{x}} &= \boldsymbol{H}^{\mathrm{t}}\boldsymbol{y} \\ \widehat{\boldsymbol{x}} &= (\boldsymbol{H}^{\mathrm{t}}\boldsymbol{H} + \mu \boldsymbol{D}^{\mathrm{t}}\boldsymbol{D})^{-1} \, \boldsymbol{H}^{\mathrm{t}}\boldsymbol{y} \\ &= \boldsymbol{F}^{\dagger}(\boldsymbol{\Lambda}_h^{\dagger}\boldsymbol{\Lambda}_h + \mu \boldsymbol{\Lambda}_d^{\dagger}\boldsymbol{\Lambda}_d)^{-1}\boldsymbol{\Lambda}_h^{\dagger}\boldsymbol{F}\boldsymbol{y} \end{aligned}$$

Computation by FFT

Huber penalty: efficient iterative computation

- Reminder: extended criterion

$$\tilde{\mathcal{J}}(\boldsymbol{x},\boldsymbol{a}) = \|\boldsymbol{y}-\boldsymbol{H}\boldsymbol{x}\|^2 + \mu \sum \frac{1}{2}\left[(x_p - x_q) - a_{pq}\right]^2 + \zeta(a_{pq})$$

1. Minimisation w.r.t. $\boldsymbol{x}$ for fixed $\boldsymbol{a}$:

$$\begin{aligned}\widetilde{\boldsymbol{x}} &= (\boldsymbol{H}^{\mathrm{t}}\boldsymbol{H} + \mu \boldsymbol{D}^{\mathrm{t}}\boldsymbol{D})^{-1}\ (\boldsymbol{H}^{\mathrm{t}}\boldsymbol{y} + \bar{\mu}\,\boldsymbol{D}^{\mathrm{t}}\boldsymbol{a}) \\ &= \boldsymbol{F}^{\dagger}\ (\boldsymbol{\Lambda}_h^{\dagger}\boldsymbol{\Lambda}_h + \mu\boldsymbol{\Lambda}_d^{\dagger}\boldsymbol{\Lambda}_d)^{-1}(\boldsymbol{\Lambda}_h^{\dagger}\boldsymbol{F}\boldsymbol{y} + \bar{\mu}\,\boldsymbol{\Lambda}_d^{\dagger}\boldsymbol{F}\boldsymbol{a})\end{aligned}$$

2. Minimisation w.r.t. $\boldsymbol{a}$ for fixed $\boldsymbol{x}$:

$$\begin{aligned}\widetilde{a}_{pq} &= \delta_{pq} - \varphi'(\delta_{pq}) \\ &= \delta_{pq}\left[1 - 2\alpha \min\left(1; s/\delta_{pq}\right)\right] \qquad \text{for Huber penalty}\end{aligned}$$

Non-quadratic and interacting

$$\rightsquigarrow \begin{cases} \text{Interacting but quadratic (and FFT)} \\ \text{Non-quadratic but non-interacting (and explicit)} \end{cases}$$

Constraints: efficient iterative computation

- Reminder: augmented Lagrangian

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{\ell}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \|\boldsymbol{D}\boldsymbol{x}\|^2 + \rho \|\boldsymbol{x} - \boldsymbol{a}\|^2 + \boldsymbol{\ell}^{\mathrm{t}}(\boldsymbol{x} - \boldsymbol{a})$$

1. Unconstrained minimisation w.r.t. $\boldsymbol{x}$

$$\begin{aligned} \widetilde{\boldsymbol{x}} &= (\boldsymbol{H}^{\mathrm{t}}\boldsymbol{H} + \mu \boldsymbol{D}^{\mathrm{t}}\boldsymbol{D} + \rho \boldsymbol{I})^{-1} \left(\boldsymbol{H}^{\mathrm{t}}\boldsymbol{y} + [\rho \boldsymbol{a} - \boldsymbol{\ell}/2]\right) \\ &= \boldsymbol{F}^{\dagger} \left(\boldsymbol{\Lambda}_h^{\dagger}\boldsymbol{\Lambda}_h + \mu \boldsymbol{\Lambda}_d^{\dagger}\boldsymbol{\Lambda}_d + \rho \boldsymbol{I}\right)^{-1} (\boldsymbol{\Lambda}_h^{\dagger}\boldsymbol{F}\boldsymbol{y} + \rho \boldsymbol{F}\boldsymbol{a} - \boldsymbol{F}\boldsymbol{\ell}/2) \end{aligned}$$

2. Constrained minimisation w.r.t. $\boldsymbol{a}$

$$\widetilde{a}_p = \begin{cases} \max\left(0, x_p + \ell_p/(2\rho)\right) & \text{for } p \in \mathcal{S} \\ 0 & \text{for } p \in \bar{\mathcal{S}} \end{cases}$$

3. Update $\boldsymbol{\ell}$

Quadratic and interacting with constraints

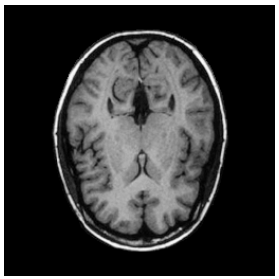
$$\rightsquigarrow \begin{cases} \text{Quadratic interacting but unconstrained (and FFT)} \\ \text{Constrained but non-interacting (and quadratic)} \end{cases}$$

Object computation / update: other possibilities

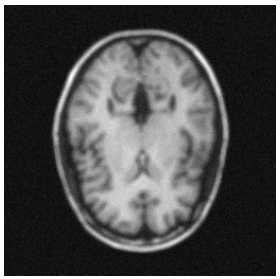
Various options and many relationships...

- Direct calculus, compact (closed) form, matrix inversion
- Algorithms for linear system
  - Gauss, Gauss-Jordan
  - Substitution
  - Triangularisation,...
- Numerical optimisation
  - Gradient descent... and various modifications
  - Pixel wise, pixel by pixel
- Diagonalization
  - Circulant approximation and diagonalization by FFT
- Special algorithms, especially for 1D case
  - Recursive least squares
  - Kalman smoother or filter (and fast versions,...)

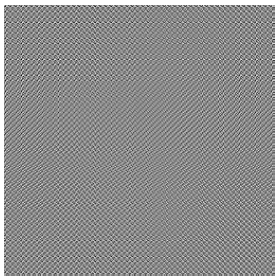
Images, deconvolution results



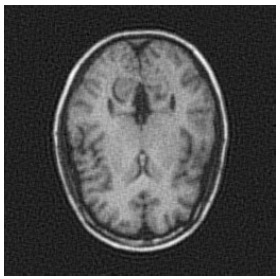
Input



Observations

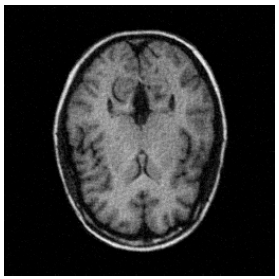


Least squares

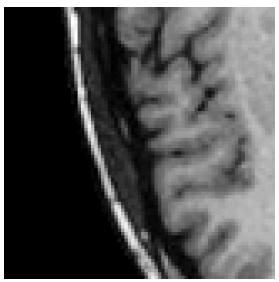


Quadratic

Huber

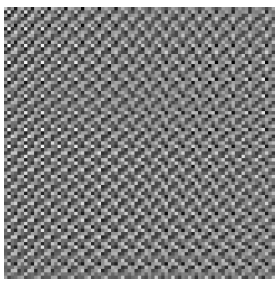


Constrained



Input

Observations



Least squares

Quadratic

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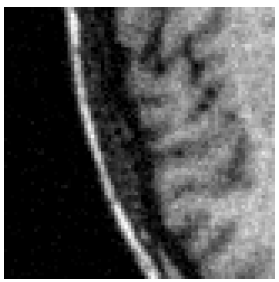

Constrained

Common structure

Iterative scheme

1. Minimisation w.r.t. object x
   - Update by FFT (weighting the Fourier domain)
2. Minimisation w.r.t. some auxiliary variables
   - Update by non-linear separable transform

...

Common structure

Iterative scheme

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Common structure

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Convolutional network

Conclusions

Synthesis: three problems / three solutions

- Quadratic penalty and smoothness (avoid exploding solution)
- Edge preserving and non-quadratic penalties
- Taking constraints into account

A first extension: a fourth problem and a solution

- Both of them [Henrot 2013]
  - Edge preserving and non-quadratic penalties
  - Taking constraints into account

. . . in a unique reconstruction method and an efficient algorithm

Conclusions

Synthesis: formal context

- Convolution and noise / Deconvolution and denoising
- Images (2D)

Extensions

- Non-invariant linear observation model
  - Tomography, scanner,...
  - Missing data
    - interpolation / inpainting
    - extrapolation / outpainting
  - Super-resolution
- Various dimensions (in each pixel)...
  - Colour images, multispectral, hyperspectral
  - Vectorial imaging (*e.g.*, speed, electromagnetic field...)
- ...and various dimensions (of the object)...
  - Signal 1D, volume 3D and more,... *e.g.*, video,...
  - 3D+t, the 3D beating heart ...

Conclusions: error and noise

Synthesis

- Additive and zero-mean and Gaussian and white and homogeneous

Extensions

- Non-homogeneous: various confidence
- Correlated noise
  - Unknown / known structure / parameters
  - Correlated to the object...
- Non-Gaussian
  - Impulsive noises: Cauchy, alpha-stable
  - Outliers and robustness
  - Counting process, *e.g.*, Poisson
- Various interaction data-noise
  - Multiplicative
  - ...other model of interaction

Conclusions: extra variables

Synthesis

- Image reconstruction, pixels only

Extensions: hidden variables, latent variables

- Object
  - Detection (singular points,...)
  - Segmentation
    - Contours
    - Regions, labels
- Noise
  - Detection of outliers and robustness
- Extra-decision
  - Make a decision regarding...

Conclusions: hyper and extra parameters

Synthesis

- Two hyperparameters: image and noise level (γ_x and γ_e)
- Bayesian interpretation and Gaussian models

Extensions

- Instrument parameter
  - Width of impulse response
  - Other parameters,...
- Object and noise : non-Gaussian models
  - Partition function...
  - Sampling itself...
- Object and noise extra-parameters
  - Labels, contours,...
  - Singular data / pixels
  - Outliers,...

Conclusions: textures, smoothness, spikes,. . .

Synthesis

- Smooth and edges

Extensions: textured image and prior / penalty

- Gaussian models
  - Oriented, quasi-periodic
  - Specific correlation function
- Advanced probabilistic models
- Dictionary and decomposition - recomposition

Conclusions: model selection / comparison

Synthesis: given model

- Instrument, direct model,...
- Adequation to the data and model for noise
- Penalty and prior model

Extensions: model selection

- Observation model
  - Gaussian *vs* Lorentzian blur
- Model for noise
  - Family: Gaussian, Cauchy,...
  - Structure: Correlated or not
- Penalty / prior model
  - Structure of penalty: neighborhood, potential,...
  - Prior model: Gauss, Huber,...
  - Number of class

Conclusions: uncertainty quantification

Synthesis

- Bias...
- Standard deviation
  - Bayesian statement: posterior standard deviation
  - Deterministic statement: Hessian and curvature (spatial or Fourier)

Extensions

- Standard deviation
  - Combination / sets / region... of pixels
- Probability of decisions
  - Detection of object
  - Label and segmentation
  - State, status,...
- Uncertainty propagation
  - Forward and inverse

Some historical landmarks

- Quadratic approaches and linear filtering ~ 60
  - Phillips, Twomey, Tikhonov
  - Kalman
  - Hunt (and Wiener ~ 40)
- Extension: discrete hidden variables ~ 80
  - Kormylo & Mendel (impulsions, peaks,...)
  - Geman & Geman (lines, contours, edges,...)
  - Besag, Graffigne, Descombes (regions, labels,...)
- Convex penalties (also hidden variables,...) ~ 90
  - $\mathrm{L}_2 - \mathrm{L}_1$, Huber, hyperbolic: Sauer, Blanc-Féraud, Idier...
  - ...et les POCS
  - L_1: Alliney-Ruzinsky, Taylor ~ 79, Yarlagadda ~ 85 ...
  - And... L_1-boom ~ 2005
- Back to more complex models ~ 2000
  - Unsupervised, myopic, semi-blind, blind
  - Stochastic sampling (MCMC, Metropolis-Hastings...)

And other imaging. . . fields, modalities, problems,. . .

Modalities

- Interferometry (radio, optical, coherent,. . .)
- Magnetic Resonance Imaging
- Tomography based on X-ray, optical wavelength, tera-Hertz,. . .
- Ultrasonic imaging, sound, mechanical
- Holography
- Polarimetry: optical and other
- Synthetic aperture radars
- Microscopy, atomic force microscopy
- Camera, photography
- Lidar, radar, sonar,. . .
- . . .

$\rightsquigarrow$ Essentially "wave $\leftrightarrow$ matter" interaction

And other imaging. . . fields, modalities, problems,. . .

Fields

- Astronomy, geology, hydrology,. . .
- Thermography, fluid mechanics, transport phenomena,. . .
- Medical: diagnosis, prognosis, theranostics,. . .
- Remote sensing, airborne imaging,. . .
- Surveillance, security,. . .
- Non destructive evaluation, control,. . .
- Computer vision, under bad conditions,. . .
- Augmented reality, computer vision & graphics,. . .
- Photography, games, recreational activities, leisure,. . .
- . . .

  $\rightsquigarrow$ Health, knowledge, leisure,. . .
  $\rightsquigarrow$ Augmented Reality, Computer Vision & Graphics,. . .
  $\rightsquigarrow$ Aerospace, aeronautics, transport, energy, industry,. . .

- Thank you for your commitment...
- ...and to my colleagues (Université & IPCV, ENSEIRB, IOGS,...)
- Visit our websites
  - `http://giovannelli.free.fr/` and also
  - `https://gretsi.fr/ressources-pedagogiques/`
- Do not hesitate: `Giova@IMS-Bordeaux.fr`