Image restoration: constrained approaches

— Support and positivity —

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Topics

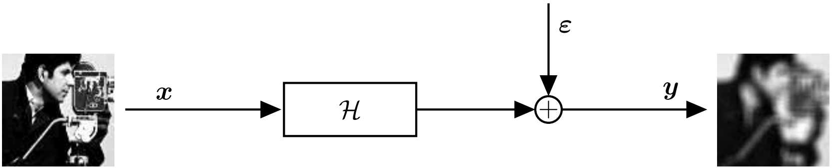
- Image restoration, deconvolution
  - Motivating examples: medical, astrophysical, industrial, vision,...
  - Various problems: deconvolution, Fourier synthesis, denoising...
  - Missing information: ill-posed character and regularisation

- Three types of regularised inversion
  1. Quadratic penalties and linear solutions
     - Closed-form expression
     - Computation through FFT
     - Optimisation (*e.g.*, gradient), system solvers (*e.g.*, splitting)
  2. Non-quadratic penalties and edge preservation
     - Half-quadratic approaches, including computation through FFT
     - Optimisation (*e.g.*, gradient), system solvers (*e.g.*, splitting)
  3. Constraints: positivity and support
     - Augmented Lagrangian and ADMM, including computation by FFT
     - Optimisation (*e.g.*, gradient), system solvers (*e.g.*, splitting)

- Bayesian strategy: a few incursions
  - Tuning hyperparameters, instrument parameters,...
  - Hidden / latent parameters, segmentation, detection,...

Convolution / Deconvolution

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{\varepsilon} = \boldsymbol{h} \star \boldsymbol{x} + \boldsymbol{\varepsilon}$$



$$\widehat{\boldsymbol{x}} = \widehat{\mathcal{X}}(\boldsymbol{y})$$

Restoration, deconvolution-denoising

- General problem: ill-posed inverse problems, *i.e., lack of information*
- Methodology: regularisation, *i.e., information compensation*
  - Specificity of the inversion / reconstruction / restoration methods
  - Trade off and tuning parameters
- Limited quality results

Regularized inversion through penalty: two terms

- Known: $\boldsymbol{H}$ and $\boldsymbol{y}$ / Unknown: $\boldsymbol{x}$
- Compare observations $\boldsymbol{y}$ and model output $\boldsymbol{H}\boldsymbol{x}$

$$J_{\text{LS}}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2$$

- Quadratic penalty of the gray level gradient
(or other linear combinations)

$$\mathcal{P}(\boldsymbol{x}) = \sum_{p \sim q} (x_p - x_q)^2 = \|\boldsymbol{D}\boldsymbol{x}\|^2$$

- Least squares and quadratic penalty:

$$\mathcal{J}_{\text{PLS}}(\boldsymbol{x}) \quad = \quad \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \, \|\boldsymbol{D}\boldsymbol{x}\|^2$$

Quadratic penalty: criterion and solution

- Least squares and quadratic penalty:

$$\mathcal{J}_{\text{PLS}}(\boldsymbol{x}) = \left\|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\right\|^2 + \mu \left\|\boldsymbol{D}\boldsymbol{x}\right\|^2$$

- Restored image

$$\begin{aligned}
\widehat{\boldsymbol{x}}_{\text{PLS}} &= \arg\min_{\boldsymbol{x}} \mathcal{J}_{\text{PLS}}(\boldsymbol{x}) \\
(\boldsymbol{H}^{\text{t}}\boldsymbol{H} + \mu\boldsymbol{D}^{\text{t}}\boldsymbol{D})\,\widehat{\boldsymbol{x}}_{\text{PLS}} &= \boldsymbol{H}^{\text{t}}\boldsymbol{y} \\
\widehat{\boldsymbol{x}}_{\text{PLS}} &= (\boldsymbol{H}^{\text{t}}\boldsymbol{H} + \mu\boldsymbol{D}^{\text{t}}\boldsymbol{D})^{-1}\,\boldsymbol{H}^{\text{t}}\boldsymbol{y}
\end{aligned}$$

- Computations based on diagonalization through FFT

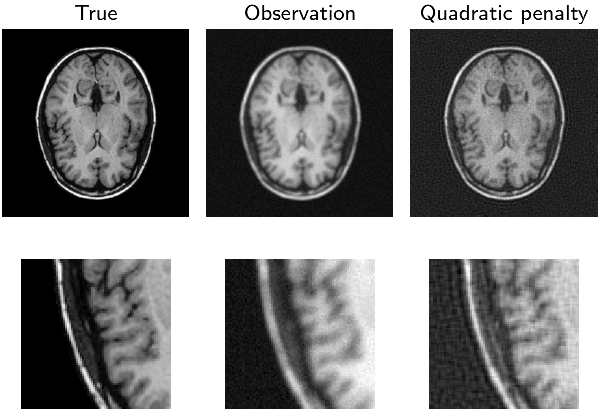
$$\begin{aligned}
\mathring{\widehat{\boldsymbol{x}}} &= (\boldsymbol{\Lambda}_h^{\dagger}\boldsymbol{\Lambda}_h + \mu\boldsymbol{\Lambda}_d^{\dagger}\boldsymbol{\Lambda}_d)^{-1}\boldsymbol{\Lambda}_h^{\dagger}\mathring{\boldsymbol{y}} \\
\mathring{\widehat{x}}_n &= \frac{\mathring{h}_n^{*}}{|\mathring{h}_n|^2 + \mu|\mathring{d}_n|^2}\,\mathring{y}_n \quad \text{for } n = 1, \ldots N
\end{aligned}$$

Object computation: other possibilities

Various options and many relationships...

- Direct calculus, compact (closed) form, matrix inversion
- Algorithms for linear system
  - Gauss, Gauss-Jordan
  - Substitution
  - Triangularisation,...
- Numerical optimisation
  - gradient descent... and various modifications
  - Pixel wise, pixel by pixel
- Diagonalization
  - Circulant approximation and diagonalization by FFT
- Special algorithms, especially for 1D case
  - Recursive least squares
  - Kalman smoother or filter (and fast versions,...)

Solution from least squares and quadratic penalty



Synthesis and extensions to constraints

- Limited capability to manage conflict between
  - Smoothing and
  - Avoiding noise explosion

  ...that limits resolution capabilities

Extension to non-quadratic penalty

- Less "smoothing" around "discontinuities"
  - Ambivalence:
    - Smoothing (homogeneous regions)
    - Heightening, enhancement, sharpening (discontinuities, edges)
  - ...and new compromise, trade off, conciliation

Another extension: include constraints

- Positivity and support
- Better physics and improved resolution
- Resort to the linear solution and FFT (Wiener-Hunt)
  - Augmented Lagrangian and ADMM

Taking constraints into account

- Expected benefits
  - Better physical modelling
  - More information $\rightsquigarrow$ "quality" improvement
  - Improved resolution

- Restoration technology
  - Still based on a penalised criterion...

$$\mathcal{J}_{\mathrm{PLS}}(\boldsymbol{x}) = \left\|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\right\|^2 + \mu \left\|\boldsymbol{D}\boldsymbol{x}\right\|^2$$

  - ...restored image still defined as a minimiser...

$$\widehat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \mathcal{J}_{\mathrm{PLS}}(\boldsymbol{x})$$

  - ...but including constraints
    ...(about the value of the gray level of pixels)

Taking constraints into account: positivity and support

- Notation
  - $\mathcal{M}$: index set of the image pixels
  - $\mathcal{S}, \mathcal{D}$: index set of a subset (support, region, mask,...) of the pixels

Investigated constraints here

- Positivity

$$\mathsf{C_p} : \forall p \in \mathcal{M} \,, \quad x_p \geqslant 0$$

- Support, mask

$$\mathsf{C_s} : \forall p \in \bar{\mathcal{S}} \,, \quad x_p = 0$$

Extensions (non investigated here)

- Template

$$\forall p \in \mathcal{M} \,, \quad t_p^- \leqslant x_p \leqslant t_p^+$$

- Partially known map

$$\forall p \in \mathcal{D} \,, \quad x_p = m_p$$

Taking constraints into account: positivity and support

General form inequality / equality

$$\boldsymbol{B}\boldsymbol{x} - \boldsymbol{b} \geqslant 0 \quad \text{et} \quad \boldsymbol{A}\boldsymbol{x} - \boldsymbol{a} = 0$$

- Positivity

$$\mathsf{C_p} : \forall p \in \mathcal{M}\,, \quad x_p \geqslant 0 \quad \rightsquigarrow \quad \boldsymbol{B} = \boldsymbol{I} \;\; \text{et} \;\; \boldsymbol{b} = \boldsymbol{0}$$

- Support

$$\mathsf{C_s} : \forall p \in \bar{\mathcal{S}}\,, \quad x_p = 0 \quad \rightsquigarrow \quad \boldsymbol{A} = \boldsymbol{T}_{\mathcal{S}} \;\; \text{et} \;\; \boldsymbol{a} = \boldsymbol{0}$$

- Template

$$\begin{aligned} \forall p \in \mathcal{M}\,, \qquad t_p^- \leqslant x_p \quad &\rightsquigarrow \quad \boldsymbol{B} = \boldsymbol{I} \;\; \text{et} \;\; \boldsymbol{b} = \boldsymbol{t}^- \\ x_p \leqslant t_p^+ \quad &\rightsquigarrow \quad \boldsymbol{B} = -\boldsymbol{I} \;\; \text{et} \;\; \boldsymbol{b} = -\boldsymbol{t}^+ \end{aligned}$$

- Partially known map

$$\forall p \in \mathcal{D}\,, \quad x_p = m_p \quad \rightsquigarrow \quad \boldsymbol{A} = \boldsymbol{T}_{\mathcal{D}} \;\; \text{et} \;\; \boldsymbol{a} = \boldsymbol{m}$$

Constrained minimiser

Theoretical point: criterion, constraint and property

- Quadratic criterion: $\mathcal{J}_{\mathrm{PLS}}(\boldsymbol{x}) = \left\|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\right\|^2 + \mu \left\|\boldsymbol{D}\boldsymbol{x}\right\|^2$
- Linear constraints: $\begin{cases} x_p = 0 & \text{for } p \in \bar{\mathcal{S}} \\ x_p \geqslant 0 & \text{for } p \in \mathcal{M} \end{cases}$
- Question of convexity
  - Convex (strict) criterion
  - Convex constraint set

Theoretical point: construction of the solution

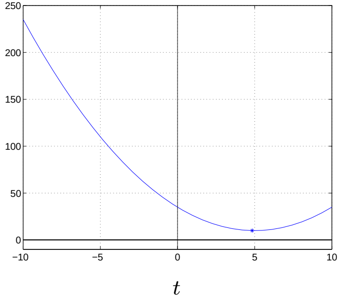
- Solution: *the only constrained minimiser*

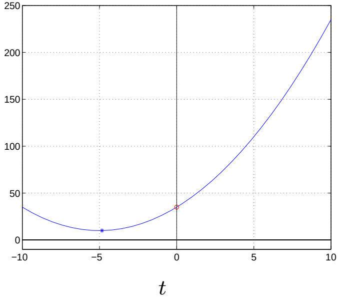
$$\widehat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \begin{cases} \left\|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\right\|^2 + \mu \left\|\boldsymbol{D}\boldsymbol{x}\right\|^2 \\ \text{s.t.} \begin{cases} x_p = 0 & \text{for } p \in \bar{\mathcal{S}} \\ x_p \geqslant 0 & \text{for } p \in \mathcal{M} \end{cases} \end{cases}$$

Constraints: some illustrations

Positivity: one variable

- One variable: $\alpha(t-\bar{t})^2+\gamma$



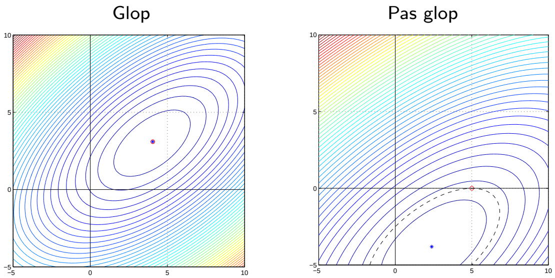


- Unconstrained solution: $\widehat{t}=\bar{t}$
- Constrained solution: $\widehat{t}=\max\,[\,0,\bar{t}\,]$

- Active and inactive constraints

Positivity: two variables (1)

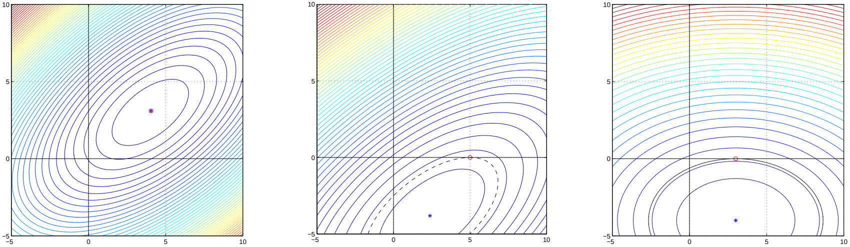
- Two variables: $\alpha_1(t_1 - \bar{t_1})^2 + \alpha_2(t_2 - \bar{t_2})^2 + \beta(t_2 - t_1)^2 + \gamma$



- Sometimes / often difficult to deduce
  - the constrained minimiser
  - from the unconstrained one

Positivity: two variables (2)

- Two variables: $\alpha_1(t_1 - \bar{t_1})^2 + \alpha_2(t_2 - \bar{t_2})^2 + \beta(t_2 - t_1)^2 + \gamma$

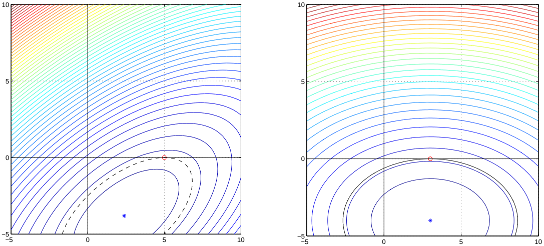


1 — 2a — 2b

- Constrained solution $=$ Unconstrained solution (1)
- Constrained solution $\neq$ Unconstrained solution (2)

  ... so active constraints

- Two variables: $\alpha_1(t_1 - \bar{t_1})^2 + \alpha_2(t_2 - \bar{t_2})^2 + \beta(t_2 - t_1)^2 + \gamma$



2a   2b

- Constrained solution $\neq$ Unconstrained solution (2)
  ... so active constraints
  - Constrained solution $\neq$ Projected unconstrained solution (2a)

$$\left(\widehat{t_1}; \widehat{t_2}\right) \neq \left(\max\left[0, \bar{t_1}\right]; \max\left[0, \bar{t_2}\right]\right)$$

  - Constrained solution $=$ Projected unconstrained solution (2b)

$$\left(\widehat{t_1}; \widehat{t_2}\right) = \left(\max\left[0, \bar{t_1}\right]; \max\left[0, \bar{t_2}\right]\right)$$

Numerical optimisation: state of the art

Problem

- Quadratic optimisation with linear constraints
- Difficulties
  - $N \sim 1\,000\,000$
  - Constraints $\oplus$ non-separable variables

Existing algorithms

- Existing tools *with guaranteed convergence* [Bertsekas 95,99; Nocedal 00,08; Boyd 04,11]
  - Gradient projection methods, constrained gradient method
  - Broyden-Fletcher-Goldfarb-Shanno (BFGS) and limited memory
  - Interior points and barrier
  - Pixel-wise descent
  - Augmented Lagrangian, ADMM
    - Constrained but separated $+$ non-separated but non-constrained
    - Partial solutions still through FFT

Equality constraints

Simplified problem

$$\widehat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \begin{cases} \left\|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\right\|^2 + \mu \left\|\boldsymbol{D}\boldsymbol{x}\right\|^2 \\ \text{s.t.} \;\; x_p = 0 \;\; \text{for } p \in \bar{\mathcal{S}} \end{cases}$$

- Sets and subsets of pixels
  - $\mathcal{M}$: full vector of pixels $\rightsquigarrow \boldsymbol{x} \in \mathbb{R}^N$
  - $\mathcal{S}$: vector of unconstrained pixels $\rightsquigarrow \bar{\boldsymbol{x}} \in \mathbb{R}^M$
- Truncation
  - $\bar{\boldsymbol{x}} = \boldsymbol{T}\boldsymbol{x}$ truncation, selection of unconstrained pixels
  - $\boldsymbol{T}$ is $M \times N$ $(M < N)$, *e.g.*, $\boldsymbol{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$
- Properties: zero-padding,...
  - $\boldsymbol{T}^{\mathrm{t}}\bar{\boldsymbol{x}}$ zero-padding, fill with zeros
  - $\boldsymbol{T}\boldsymbol{T}^{\mathrm{t}} = \boldsymbol{I}_M$
  - $\boldsymbol{T}^{\mathrm{t}}\boldsymbol{T} = \mathrm{diag}[\,\ldots 0\,/\,1 \ldots\,]$: projection, "nullification matrix"

Equality: direct closed form expression

- Original (unconstrained) criterion

$$\mathcal{J}_{\text{PLS}}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \|\boldsymbol{D}\boldsymbol{x}\|^2$$

- Zero-padded variable

$$\boldsymbol{x} = \boldsymbol{T}^{\text{t}} \bar{\boldsymbol{x}}$$

- Restricted criterion

$$\bar{\mathcal{J}}_{\text{PLS}}(\bar{\boldsymbol{x}}) = \left\|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{T}^{\text{t}}\bar{\boldsymbol{x}}\right\|^2 + \mu \left\|\boldsymbol{D}\boldsymbol{T}^{\text{t}}\bar{\boldsymbol{x}}\right\|^2$$

- Closed form expression for the solution

$$\begin{aligned}
\widehat{\bar{\boldsymbol{x}}} &= \underset{\bar{\boldsymbol{x}} \in \mathbb{R}^M}{\arg\min}\, \bar{\mathcal{J}}_{\text{PLS}}(\bar{\boldsymbol{x}}) \\
&= \left[\boldsymbol{T}\boldsymbol{H}^{\text{t}}\boldsymbol{H}\boldsymbol{T}^{\text{t}} + \mu \boldsymbol{T}\boldsymbol{D}^{\text{t}}\boldsymbol{D}\boldsymbol{T}^{\text{t}}\right]^{-1} \boldsymbol{T}\boldsymbol{H}^{\text{t}}\boldsymbol{y} \\
&= \left[\boldsymbol{T}\left(\boldsymbol{H}^{\text{t}}\boldsymbol{H} + \mu \boldsymbol{D}^{\text{t}}\boldsymbol{D}\right)\boldsymbol{T}^{\text{t}}\right]^{-1} \boldsymbol{T}\boldsymbol{H}^{\text{t}}\boldsymbol{y}
\end{aligned}$$

$$\begin{aligned}
\widehat{\boldsymbol{x}} &= \boldsymbol{T}^{\text{t}}\, \bar{\boldsymbol{x}} \\
&= \boldsymbol{T}^{\text{t}} \left[\boldsymbol{T}\left(\boldsymbol{H}^{\text{t}}\boldsymbol{H} + \mu \boldsymbol{D}^{\text{t}}\boldsymbol{D}\right)\boldsymbol{T}^{\text{t}}\right]^{-1} \boldsymbol{T}\boldsymbol{H}^{\text{t}}\boldsymbol{y}
\end{aligned}$$

Equality: closed form expression via Lagrangian

- Original (unconstrained) criterion

$$\mathcal{J}_{\text{PLS}}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \|\boldsymbol{D}\boldsymbol{x}\|^2$$

- Equality constraints:

$$\begin{aligned} x_p &= 0 \text{ for } p \in \bar{\mathcal{S}} \\ \bar{\boldsymbol{T}}\boldsymbol{x} &= \boldsymbol{0} \end{aligned}$$

- Equality constraints and Lagrangian term

$$\sum_{p \in \bar{\mathcal{S}}} \ell_p x_p = \boldsymbol{\ell}^{\text{t}} \bar{\boldsymbol{T}}\boldsymbol{x}$$

- Lagrangian

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\ell}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \|\boldsymbol{D}\boldsymbol{x}\|^2 + \boldsymbol{\ell}^{\text{t}} \bar{\boldsymbol{T}}\boldsymbol{x}$$

- Closed form expression (see exercise)

$$\widehat{\boldsymbol{x}} = \left[\boldsymbol{Q}^{-1} - \boldsymbol{Q}^{-1} \bar{\boldsymbol{T}}^{\text{t}} (\bar{\boldsymbol{T}}\boldsymbol{Q}^{-1}\bar{\boldsymbol{T}}^{\text{t}})^{-1} \bar{\boldsymbol{T}}\boldsymbol{Q}^{-1} \right] \boldsymbol{H}^{\text{t}}\boldsymbol{y}$$

$$\boldsymbol{Q} = (\boldsymbol{H}^{\text{t}}\boldsymbol{H} + \mu\boldsymbol{D}^{\text{t}}\boldsymbol{D})$$

Equality: practical algorithm via Lagrangian

- Original (unconstrained) criterion

$$\mathcal{J}_{\text{PLS}}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \|\boldsymbol{D}\boldsymbol{x}\|^2$$

- Equality constraints:

$$\bar{\boldsymbol{T}}\boldsymbol{x} = 0$$

- Lagrangian

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\ell}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \|\boldsymbol{D}\boldsymbol{x}\|^2 + \boldsymbol{\ell}^{\mathrm{t}}\bar{\boldsymbol{T}}\boldsymbol{x}$$

- Iterative algorithm

$$\begin{cases} \boldsymbol{x}^{[k+1]} & = \ \underset{\boldsymbol{x}}{\arg\min}\, \mathcal{L}(\boldsymbol{x}, \boldsymbol{\ell}^{[k]}) \ = \ (\boldsymbol{H}^{\mathrm{t}}\boldsymbol{H} + \mu \boldsymbol{D}^{\mathrm{t}}\boldsymbol{D})^{-1}(\boldsymbol{H}^{\mathrm{t}}\boldsymbol{y} - \bullet) \\ \boldsymbol{\ell}^{[k+1]} & = \ \boldsymbol{\ell}^{[k]} + \tau_k \, \bar{\boldsymbol{T}}\boldsymbol{x}^{[k+1]} \end{cases}$$

Equality: practical algorithm via Lagrangian

- Original (unconstrained) criterion

$$\mathcal{J}_{\text{PLS}}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \|\boldsymbol{D}\boldsymbol{x}\|^2$$

- Equality constraints:

$$\bar{\boldsymbol{T}}\boldsymbol{x} = 0$$

- Lagrangian

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\ell}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \|\boldsymbol{D}\boldsymbol{x}\|^2 + \boldsymbol{\ell}^{\mathrm{t}}\bar{\boldsymbol{T}}\boldsymbol{x}$$

- Iterative algorithm

$$\begin{cases} \boldsymbol{x}^{[k+1]} & = \arg\min_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\ell}^{[k]}) = (\boldsymbol{H}^{\mathrm{t}}\boldsymbol{H} + \mu \boldsymbol{D}^{\mathrm{t}}\boldsymbol{D})^{-1}(\boldsymbol{H}^{\mathrm{t}}\boldsymbol{y} - \bar{\boldsymbol{T}}^{\mathrm{t}}\boldsymbol{\ell}^{[k]}/2) \\ \boldsymbol{\ell}^{[k+1]} & = \boldsymbol{\ell}^{[k]} + \tau_k \, \bar{\boldsymbol{T}}\boldsymbol{x}^{[k+1]} \end{cases}$$

Equality: algorithm via augmented Lagrangian

- Original (unconstrained) criterion

$$\mathcal{J}^{+}_{\text{PLS}}(\boldsymbol{x}) = \left\|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\right\|^2 + \mu \left\|\boldsymbol{D}\boldsymbol{x}\right\|^2 + \rho \left\|\bar{\boldsymbol{T}}\boldsymbol{x}\right\|^2$$

- Equality constraints:

$$\bar{\boldsymbol{T}}\boldsymbol{x} = 0$$

- Lagrangian

$$\mathcal{L}_\rho(\boldsymbol{x}, \boldsymbol{\ell}) = \left\|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\right\|^2 + \mu \left\|\boldsymbol{D}\boldsymbol{x}\right\|^2 + \rho \left\|\bar{\boldsymbol{T}}\boldsymbol{x}\right\|^2 + \boldsymbol{\ell}^{\mathrm{t}}\bar{\boldsymbol{T}}\boldsymbol{x}$$

- Iterative algorithm

$$\begin{cases} \boldsymbol{x}^{[k+1]} & = \; (\boldsymbol{H}^{\mathrm{t}}\boldsymbol{H} + \mu\boldsymbol{D}^{\mathrm{t}}\boldsymbol{D} + \; \bullet \;)^{-1}(\boldsymbol{H}^{\mathrm{t}}\boldsymbol{y} - \bar{\boldsymbol{T}}^{\mathrm{t}}\boldsymbol{\ell}^{[k]}/2) \\ \boldsymbol{\ell}^{[k+1]} & = \; \boldsymbol{\ell}^{[k]} + 2\rho\, \bar{\boldsymbol{T}}\boldsymbol{x}^{[k+1]} \end{cases}$$

Equality: algorithm via augmented Lagrangian

- Original (unconstrained) criterion

$$\mathcal{J}^{+}_{\text{PLS}}(\boldsymbol{x}) = \left\|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\right\|^2 + \mu \left\|\boldsymbol{D}\boldsymbol{x}\right\|^2 + \rho \left\|\bar{\boldsymbol{T}}\boldsymbol{x}\right\|^2$$

- Equality constraints:

$$\bar{\boldsymbol{T}}\boldsymbol{x} = 0$$

- Lagrangian

$$\mathcal{L}_\rho(\boldsymbol{x}, \boldsymbol{\ell}) = \left\|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\right\|^2 + \mu \left\|\boldsymbol{D}\boldsymbol{x}\right\|^2 + \rho \left\|\bar{\boldsymbol{T}}\boldsymbol{x}\right\|^2 + \boldsymbol{\ell}^{\text{t}}\bar{\boldsymbol{T}}\boldsymbol{x}$$

- Iterative algorithm

$$\begin{cases} \boldsymbol{x}^{[k+1]} & = \; (\boldsymbol{H}^{\text{t}}\boldsymbol{H} + \mu\boldsymbol{D}^{\text{t}}\boldsymbol{D} + \rho\boldsymbol{T}^{\text{t}}\boldsymbol{T})^{-1}(\boldsymbol{H}^{\text{t}}\boldsymbol{y} - \bar{\boldsymbol{T}}^{\text{t}}\boldsymbol{\ell}^{[k]}/2) \\ \boldsymbol{\ell}^{[k+1]} & = \; \boldsymbol{\ell}^{[k]} + 2\rho\, \bar{\boldsymbol{T}}\boldsymbol{x}^{[k+1]} \end{cases}$$

Equality: via augmented Lagrangian and slack variables

- Original (unconstrained) criterion

$$\mathcal{J}_{\mathrm{PLS}}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \|\boldsymbol{D}\boldsymbol{x}\|^2$$

- Constraint $\oplus$ auxiliary (slack) variables

$$x_p = 0 \;\text{ for } p \in \bar{\mathcal{S}} \quad \rightsquigarrow \quad \begin{cases} x_p = s_p & \text{for } p \in \mathcal{M} \\ s_p = 0 & \text{for } p \in \bar{\mathcal{S}} \end{cases}$$

- Augmented Lagrangian $\oplus$ slack variables

$$\mathcal{L}_\rho(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{\ell}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \|\boldsymbol{D}\boldsymbol{x}\|^2 + \rho \|\boldsymbol{x} - \boldsymbol{s}\|^2 + \boldsymbol{\ell}^{\mathrm{t}}(\boldsymbol{x} - \boldsymbol{s})$$

- Iterative algorithm

$$\begin{cases} \boldsymbol{x}^{[k+1]} & = \; (\boldsymbol{H}^{\mathrm{t}}\boldsymbol{H} + \mu\boldsymbol{D}^{\mathrm{t}}\boldsymbol{D} + \rho\boldsymbol{I})^{-1}(\boldsymbol{H}^{\mathrm{t}}\boldsymbol{y} - \boldsymbol{\ell}^{[k]}/2 + \bullet) \\ s_p^{[k+1]} & = \begin{cases} \bullet & \text{for } p \in \mathcal{S} \\ 0 & \text{for } p \in \bar{\mathcal{S}} \end{cases} \\ \boldsymbol{\ell}^{[k+1]} & = \; \boldsymbol{\ell}^{[k]} + 2\rho\,(\boldsymbol{x}^{[k+1]} - \boldsymbol{s}^{[k+1]}) \end{cases}$$

Equality: via augmented Lagrangian and slack variables

- Original (unconstrained) criterion

$$\mathcal{J}_{\text{PLS}}(\boldsymbol{x}) = \left\|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\right\|^2 + \mu\left\|\boldsymbol{D}\boldsymbol{x}\right\|^2$$

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- Augmented Lagrangian $\oplus$ slack variables

$$\mathcal{L}_\rho(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{\ell}) = \left\|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\right\|^2 + \mu\left\|\boldsymbol{D}\boldsymbol{x}\right\|^2 + \rho\left\|\boldsymbol{x} - \boldsymbol{s}\right\|^2 + \boldsymbol{\ell}^{\mathrm{t}}(\boldsymbol{x} - \boldsymbol{s})$$

- Iterative algorithm

$$\begin{cases} \boldsymbol{x}^{[k+1]} & = & (\boldsymbol{H}^{\mathrm{t}}\boldsymbol{H} + \mu\boldsymbol{D}^{\mathrm{t}}\boldsymbol{D} + \rho\boldsymbol{I})^{-1}(\boldsymbol{H}^{\mathrm{t}}\boldsymbol{y} - \boldsymbol{\ell}^{[k]}/2 + \rho\boldsymbol{s}^{[k]}) \\ s_p^{[k+1]} & = & \begin{cases} \bullet & \text{for } p \in \mathcal{S} \\ 0 & \text{for } p \in \bar{\mathcal{S}} \end{cases} \\ \boldsymbol{\ell}^{[k+1]} & = & \boldsymbol{\ell}^{[k]} + 2\rho\left(\boldsymbol{x}^{[k+1]} - \boldsymbol{s}^{[k+1]}\right) \end{cases}$$

Equality: via augmented Lagrangian and slack variables

- Original (unconstrained) criterion

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- Constraint $\oplus$ auxiliary (slack) variables

$$x_p = 0 \;\; \text{for } p \in \bar{\mathcal{S}} \quad \rightsquigarrow \quad \begin{cases} x_p = s_p & \text{for } p \in \mathcal{M} \\ s_p = 0 & \text{for } p \in \bar{\mathcal{S}} \end{cases}$$

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- Iterative algorithm

$$\begin{cases} \boldsymbol{x}^{[k+1]} & = & (\boldsymbol{H}^{\text{t}}\boldsymbol{H} + \mu\boldsymbol{D}^{\text{t}}\boldsymbol{D} + \rho\boldsymbol{I})^{-1}(\boldsymbol{H}^{\text{t}}\boldsymbol{y} - \boldsymbol{\ell}^{[k]}/2 + \rho\boldsymbol{s}^{[k]}) \\ s_p^{[k+1]} & = & \begin{cases} x_p^{[k+1]} + \ell_p^{[k]}/(2\rho) & \text{for } p \in \mathcal{S} \\ 0 & \text{for } p \in \bar{\mathcal{S}} \end{cases} \\ \boldsymbol{\ell}^{[k+1]} & = & \boldsymbol{\ell}^{[k]} + 2\rho\left(\boldsymbol{x}^{[k+1]} - \boldsymbol{s}^{[k+1]}\right) \end{cases}$$

Equality and inequality constraints: problem

- Original (unconstrained) criterion

$$\mathcal{J}_{\text{PLS}}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \|\boldsymbol{D}\boldsymbol{x}\|^2$$

- Equality and inequality constraints

$$\begin{cases} x_p = 0 & \text{for } p \in \bar{\mathcal{S}} \\ x_p \geq 0 & \text{for } p \in \mathcal{M} \end{cases}$$

- Equality and inequality constraints $\oplus$ slack variables

$$\left\{ \begin{array}{l} x_p = s_p \quad \text{for } p \in \mathcal{M} \\ \begin{cases} s_p = 0 & \text{for } p \in \bar{\mathcal{S}} \\ s_p \geqslant 0 & \text{for } p \in \mathcal{M} \end{cases} \end{array} \right.$$

- Augmented Lagrangian $\oplus$ slack variables

$$\mathcal{L}_\rho(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{\ell}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \|\boldsymbol{D}\boldsymbol{x}\|^2 + \rho \|\boldsymbol{x} - \boldsymbol{s}\|^2 + \boldsymbol{\ell}^{\text{t}}(\boldsymbol{x} - \boldsymbol{s})$$

Iterative algorithm: ADMM

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{\ell}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \|\boldsymbol{D}\boldsymbol{x}\|^2 + \rho \|\boldsymbol{x} - \boldsymbol{s}\|^2 + \boldsymbol{\ell}^{\mathrm{t}}(\boldsymbol{x} - \boldsymbol{s})$$

- Iterate three steps

1. Unconstrained minimisation w.r.t. $\boldsymbol{x}$

$$\widetilde{\boldsymbol{x}} = \left(\boldsymbol{H}^{\mathrm{t}}\boldsymbol{H} + \mu\boldsymbol{D}^{\mathrm{t}}\boldsymbol{D} + \rho\boldsymbol{I}\right)^{-1}\left(\boldsymbol{H}^{\mathrm{t}}\boldsymbol{y} + [\rho\boldsymbol{s} - \boldsymbol{\ell}/2]\right) \qquad (\equiv FFT)$$

2. Constrained minimisation w.r.t. $\boldsymbol{s}$ (s.t. $s_p \geqslant 0$ or $s_p = 0$)

$$\widetilde{s}_p = \begin{cases} \max\left(0, x_p + \ell_p/(2\rho)\right) & \text{for } p \in \mathcal{S} \\ 0 & \text{for } p \in \bar{\mathcal{S}} \end{cases}$$

3. Update $\boldsymbol{\ell}$

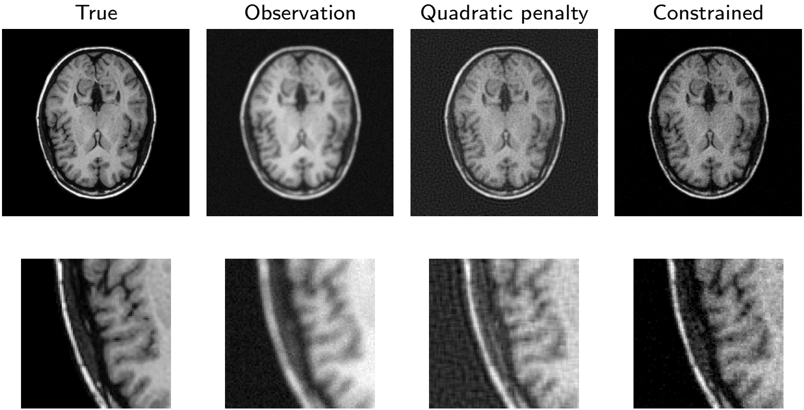
$$\widetilde{\ell}_p = \ell_p + 2\rho(x_p - s_p)$$

Object update: other possibilities

Various options and many relationship...

- Direct calculus, closed-form expression, matrix inversion
- Algorithm for linear systems
  - Gauss, Gauss-Jordan
  - Substitution
  - Triangularisation,...
- Numerical optimisation
  - Gradient descent... and modified versions
  - Pixel wise, pixel by pixel
- Diagonalization
  - Circulant approximation and diagonalization by FFT
- Special algorithms, especially for 1D case
  - Recursive least squares
  - Kalman smoother or filter (and fast versions)

Constrained solution



Conclusions

Synthesis

- Image deconvolution
- Taking constraints into account
  - Positivity and support
  - Quadratic penalty
- Numerical computations: augmented Lagrangian and ADMM
  - Iterative: quadratic $\oplus$ separable
    - Circulant case (diagonalization) $\rightsquigarrow$ FFT only
      (or numerical optimisation, system solvers,. . .)
    - Parallel (separable and explicit)

Extensions (not developped)

- Also available for
  - non-invariant linear direct model
  - colour images, multispectral and hyperspectral
  - also signal, 3D and more, video, 3D+t. . .
- Including both Huber penalty and constraints
- Hyperparameters estimation, instrument parameter estimation,. . .