Image restoration: edge preserving

— Convex penalties —

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Topics

- Image restoration, deconvolution
  - Motivating examples: medical, astrophysical, industrial, vision,...
  - Various problems: deconvolution, Fourier synthesis, denoising...
  - Missing information: ill-posed character and regularisation

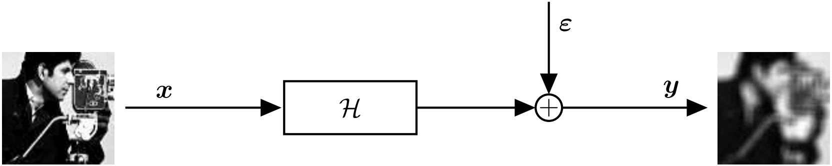
- Three types of regularised inversion

  1. Quadratic penalties and linear solutions
     - Closed-form expression
     - Computation through FFT
     - Optimisation (*e.g.*, gradient), system solvers (*e.g.*, splitting)
  2. Non-quadratic penalties and edge preservation
     - Half-quadratic approaches, including computation through FFT
     - Optimisation (*e.g.*, gradient), system solvers (*e.g.*, splitting)
  3. Constraints: positivity and support
     - Augmented Lagrangian and ADMM, including computation by FFT
     - Optimisation (*e.g.*, gradient), system solvers (*e.g.*, splitting)

- Bayesian strategy: a few incursions
  - Tuning hyperparameters, instrument parameters,...
  - Hidden / latent parameters, segmentation, detection,...

Convolution / Deconvolution

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{\varepsilon} = \boldsymbol{h} \star \boldsymbol{x} + \boldsymbol{\varepsilon}$$



$$\widehat{\boldsymbol{x}} = \widehat{\mathcal{X}}(\boldsymbol{y})$$

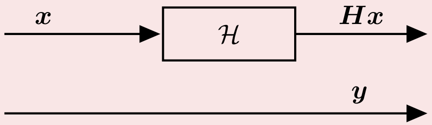
Restoration, deconvolution-denoising

- General problem: ill-posed inverse problems, *i.e., lack of information*
- Methodology: regularisation, *i.e., information compensation*
  - Specificity of the inversion / reconstruction / restoration methods
  - Trade off and tuning parameters
- Limited quality results

Competition: Adequation to data

- Compare observations $\boldsymbol{y}$ and model output $\boldsymbol{Hx}$
  - Unknown: $\boldsymbol{x}$
  - Known: $\boldsymbol{H}$ and $\boldsymbol{y}$

Comparison experiment – model



- Quadratic criterion: distance observation – model output

$$\mathcal{J}_{\mathrm{LS}}(\boldsymbol{x}) = \left\|\boldsymbol{y} - \boldsymbol{Hx}\right\|^2$$

Competition: Smoothness prior

- Data insufficiently informative
  $\rightsquigarrow$ Account for prior information
  $\rightsquigarrow$ Here: *smoothness* of images

- Quadratic penalty of the gray level "gradient"

$$\begin{aligned} \mathcal{P}(\boldsymbol{x}) &= \sum_{p\sim q} (x_p - x_q)^2 \\ &= \|\boldsymbol{D}\boldsymbol{x}\|^2 \end{aligned}$$

Quadratic penalty: criterion and solution

- Least squares and quadratic penalty:

$$J_{\text{PLS}}(\boldsymbol{x}) = \left\|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\right\|^2 + \mu \left\|\boldsymbol{D}\boldsymbol{x}\right\|^2$$

- Restored image

$$\begin{aligned}
\widehat{\boldsymbol{x}}_{\text{PLS}} &= \arg\min_{\boldsymbol{x}} J_{\text{PLS}}(\boldsymbol{x}) \\
\left(\boldsymbol{H}^{\text{t}}\boldsymbol{H} + \mu \boldsymbol{D}^{\text{t}}\boldsymbol{D}\right) \widehat{\boldsymbol{x}}_{\text{PLS}} &= \boldsymbol{H}^{\text{t}}\boldsymbol{y} \\
\widehat{\boldsymbol{x}}_{\text{PLS}} &= \left(\boldsymbol{H}^{\text{t}}\boldsymbol{H} + \mu \boldsymbol{D}^{\text{t}}\boldsymbol{D}\right)^{-1} \boldsymbol{H}^{\text{t}}\boldsymbol{y}
\end{aligned}$$

- Computations based on diagonalization through FFT

$$\begin{aligned}
\mathring{\widehat{\boldsymbol{x}}} &= \left(\boldsymbol{\Lambda}_h^{\dagger}\boldsymbol{\Lambda}_h + \mu \boldsymbol{\Lambda}_d^{\dagger}\boldsymbol{\Lambda}_d\right)^{-1} \boldsymbol{\Lambda}_h^{\dagger}\mathring{\boldsymbol{y}} \\
\mathring{\widehat{x}}_n &= \frac{\mathring{h}_n^{*}}{|\mathring{h}_n|^2 + \mu |\mathring{d}_n|^2}\, \mathring{y}_n \quad \text{for } n = 1, \dots N
\end{aligned}$$

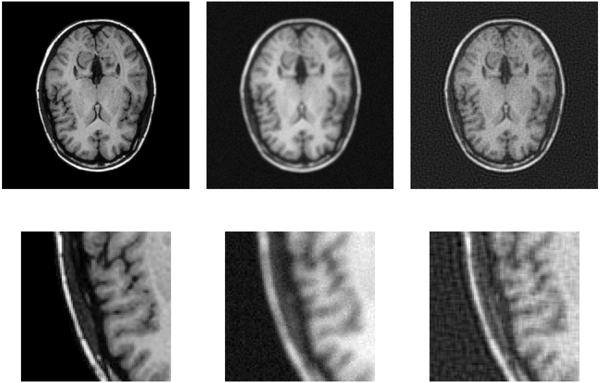
Object computation: other possibilities

Various options and many relationships...

- Direct calculus, compact (closed) form, matrix inversion
- Algorithms for linear system
  - Gauss, Gauss-Jordan
  - Substitution
  - Triangularisation,...
- Numerical optimisation
  - gradient descent... and various modifications
  - Pixel wise, pixel by pixel
- Diagonalization
  - Circulant approximation and diagonalization by FFT
- Special algorithms, especially for 1D case
  - Recursive least squares
  - Kalman smoother or filter (and fast versions,...)

Solution from least squares and quadratic penalty

True | Observation | Quadratic penalty



Synthesis and extensions to edge preservation

- Limited capability to manage conflict between
  - Smoothing and
  - Avoiding noise explosion

  . . . that limits resolution capabilities

Extension: new penalty

- Desirable: less "smoothing" around "discontinuities"
  - Ambivalence:
    - Smoothing (homogeneous regions)
    - Heightening, enhancement, sharpening (discontinuities, edges)
  - . . . and new compromise, trade off, conciliation
- Resort to the linear solution and FFT (Wiener-Hunt)

Edge preservation and non-quadratic penalties

- Restored image still defined as the minimiser...

$$\widehat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \mathcal{J}(\boldsymbol{x})$$

- ... of a penalised criterion ...

$$\mathcal{J}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu\,\mathcal{P}(\boldsymbol{x})$$

- ... once again penalising variations

$$\mathcal{P}(\boldsymbol{x}) = \sum_{p\sim q} \varphi(x_p - x_q)$$

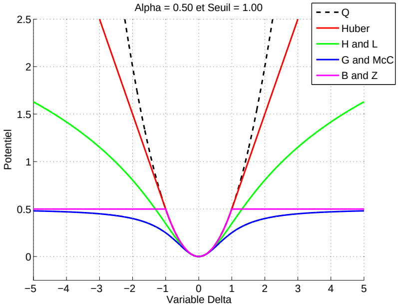
- ... but strong penalisation of "small variations"
  and less penalisation for "discontinuities"

$$\varphi(\delta) = \delta^2 \quad \rightsquigarrow \quad \varphi(\delta) = \ldots\,?$$

  - Ambivalence: new compromise, trade off, conciliation
    - Smoothing (homogeneous regions)
    - Heightening, enhancement, sharpening (discontinuities, edges)

Typical potentials φ

- Again $\varphi(\delta) \sim \delta^2$ for small δ
- Behaviour for large δ
  1. **Horizontal asymptote**
     [Blake and Zisserman (87), Geman and McClure (87)]
  2. **Horizontal parabolic behaviour**
     [Hebert and Leahy (89)]
  3. **Oblique (slant) asymptote**
     [Huber (81)]
  4. **Vertical parabolic behaviour**
     Wiener-Tikhonov solution



Four major types of potentials

1. **Horizontal asymptote** $\varphi(\delta) \sim 1$

$$\varphi(\delta) = \begin{cases} \delta^2 & \text{if } |\,\delta\,| \leqslant s \\ s^2 & \text{if } |\,\delta\,| \geqslant s \end{cases} \quad ; \quad \varphi(\delta) = s^2 \frac{(\delta/s)^2}{1 + (\delta/s)^2}$$

2. **Horizontal parabolic behaviour** $\varphi(\delta) \sim \log|\delta|$

$$\varphi(\delta) = s^2 \log\left[1 + (\delta/s)^2\right]$$

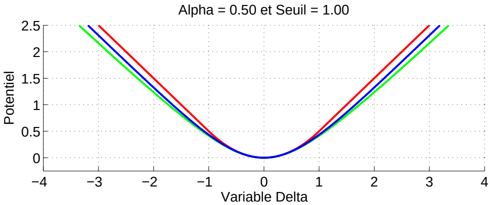
3. **Oblique (slant) asymptote** $\varphi(\delta) \sim |\delta|$

$$\varphi(\delta) = \begin{cases} \delta^2 & \text{if } |\,\delta\,| \leqslant s \\ 2s\,|\,\delta\,| - s^2 & \text{if } |\,\delta\,| \geqslant s \end{cases} \quad ; \quad \varphi(\delta) = 2s^2 \left(\sqrt{1 + [\delta/s]^2} - 1\right)$$

4. **Vertical parabolic behaviour** $\varphi(\delta) \sim \delta^2$

$$\varphi(\delta) = \delta^2$$

Potentials with oblique asymptote (L_2 / L_1): details



$$Huber: \quad \varphi(\delta) = s^2 \begin{cases} [\delta/s]^2 & \text{if } |\,\delta\,| \leqslant s \\ 2\,|\,\delta\,|/s - 1 & \text{if } |\,\delta\,| \geqslant s \end{cases}$$

$$Hyperbolic: \quad \varphi(\delta) = 2s^2 \left(\sqrt{1 + [\delta/s]^2} - 1 \right)$$

$$LogCosh: \quad \varphi(\delta) = 2s^2 \log\cosh\left(|\,\delta\,|/s\right)$$

$$FairFunction: \quad \varphi(\delta) = 2s^2 \left[\, |\,\delta\,|/s - \log\left(1 + |\,\delta\,|/s\right) \,\right]$$

More general non-quadratic penalties (1D)

- Differences, derivative and higher order, generalizations,...

$$
\begin{aligned}
\mathcal{P}(\boldsymbol{x}) &= \sum_n \varphi(x_{n+1} - x_n) \\
\mathcal{P}(\boldsymbol{x}) &= \sum_n \varphi(x_{n+1} - 2x_n + x_{n-1}) \\
\mathcal{P}(\boldsymbol{x}) &= \sum_n \varphi(\alpha\ x_{n+1} - x_n + \alpha'\ x_{n-1}) \\
\mathcal{P}(\boldsymbol{x}) &= \sum_n \varphi(\boldsymbol{\alpha}_n^{\mathrm{t}}\, \boldsymbol{x})
\end{aligned}
$$

- Linear combinations (wavelet, other-stuff-in-'et',... dictionaries,...)

$$
\mathcal{P}(\boldsymbol{x}) = \sum_n \varphi(\boldsymbol{w}_n^{\mathrm{t}} \boldsymbol{x}) = \sum_n \varphi\left(\sum_m w_{nm} x_m\right)
$$

  - Redundant or not
  - Link with Haar wavelet and other

More general non-quadratic penalties (2D)

- Differences, derivatives and higher order, gradient, generalizations

$$\begin{aligned}\mathcal{P}(\boldsymbol{x}) &= \sum_{p\sim q} \varphi(x_p - x_q) \\ &= \sum_{n,m} \varphi(x_{n+1,m} - x_{n,m}) + \sum_{n,m} \varphi(x_{n,m+1} - x_{n,m})\end{aligned}$$

  - Notion of neighborhood and Markov field
  - Any highpass filter, contour detector (Prewitt, Sobel,...)
  - Linear combinations: wavelet, contourlet and other-stuff-in-'et',...

- Other possibilities (slightly different)
  - Enforcement towards a known shape $\bar{x}$

$$\mathcal{P}(\boldsymbol{x}) = \sum_{p} \varphi(x_p - \bar{x}_p)$$

  - Separable penalty

$$\mathcal{P}(\boldsymbol{x}) = \sum_{p} \varphi(x_p)$$

Penalised least squares solution

- A reminder of the criterion and the restored image

$$\mathcal{J}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \sum_{p \sim q} \varphi_s(x_p - x_q)$$

$$\widehat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \mathcal{J}(\boldsymbol{x})$$

  - with φ_s one of the mentioned (non-quadratic) potentials
  - and two hyperparameters: μ and s

- Non-quadratic criterion
  - Non-linear gradient
  - No closed-form expression

- Two questions
  - Practical computation: numerical optimisation algorithm,...
  - Minimiser: existence, uniqueness,... continuity

Convexity and existence-uniqueness

- Convex set
  - $\mathbb{R}^N$, $\mathbb{R}_+^N$, intervals of $\mathbb{R}^N$,...
  - Properties: intersection, convex envelope, projection,...
- Strictly convex criterion, convex criterion,
  - $\Theta(u) = u^2$, $\Theta(\boldsymbol{u}) = \|\boldsymbol{u}\|^2$, $\Theta(u) = |\,u\,|$, Huber,...
  - Properties: sum of convex function, level sets,...
- Key result
  - Set of minimisers of convex criterion on a convex set is a convex set
  - Strict convexity $\rightsquigarrow$ unique minimiser
- Application
  - φ convex $\rightsquigarrow$ J convex
- In the following developments, potential φ_s
  - Huber or hyperbolic: convex (strict) $\rightsquigarrow$ guarantees
  - In addition: non-convex $\rightsquigarrow$ no guarantee (although, sometimes...)

Half-quadratic enchantement (start)

- Reminder of the criterion

$$\mathcal{J}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \sum_{p\sim q} \varphi(x_p - x_q)$$

- Minimisation based on quadratic
  - Original idea of [Geman + Yang, 95]
  - Set of auxiliary variables a_{pq} so that: $\varphi(\delta_{pq}) \quad \longleftrightarrow \quad \delta_{pq}^2$

$$\varphi(\delta) = \inf_a \left[\frac{1}{2}(\delta - a)^2 + \zeta(a) \right]$$

  - With appropriate $\zeta(a)$

- Extended criterion

$$\tilde{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{a}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \sum_{p\sim q} \frac{1}{2} \left[(x_p - x_q) - a_{pq}\right]^2 + \zeta(a_{pq})$$

und natürlich:

$$\mathcal{J}(\boldsymbol{x}) = \inf_{\boldsymbol{a}} \tilde{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{a})$$

Legendre Transform (LT) or Convex Conjugate (CC)

Definition of LT or CC (far more general than that version)

Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$

- strictly convex
- once (or twice) differentiable

The LT or CC is the function $f^\star : \mathbb{R} \longrightarrow \mathbb{R}$ defined by:

$$f^\star(t) \;=\; \sup_{x \in \mathbb{R}} \left[\, xt - f(x) \,\right]$$

Remark

$$f^\star(0) \;=\; \sup_{x \in \mathbb{R}} \left[\, -f(x) \,\right] = -\inf_{x \in \mathbb{R}} \left[\, f(x) \,\right]$$

$$\forall t, x \in \mathbb{R}, \quad xt - f(x) \leqslant f^\star(t)$$

$$\forall t, x \in \mathbb{R}, \quad f^\star(t) + f(x) \geqslant xt$$

LT: some shift and dilatation / contraction properties

$$f^\star(t) \;=\; \sup_{x\in\mathbb{R}} \;[\, xt - f(x)\,]$$

Horizontal: dilatation ($\gamma \in \mathbb{R}_+^\star$) and shift ($x_0 \in \mathbb{R}$)

$$\begin{cases} g(x) = f(\gamma x) \\ g^\star(t) = f^\star(t/\gamma) \end{cases} \qquad\qquad \begin{cases} g(x) = f(x - x_0) \\ g^\star(t) = f^\star(t) - x_0 t \end{cases}$$

Vertical: shift-dilatation ($\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_+^\star$)

$$\begin{cases} g(x) = \alpha + \beta\, f(x) \\ g^\star(t) = \beta\, f^\star(t/\beta) - \alpha \end{cases}$$

Specific case

$$\alpha = 0, \beta = 1 \quad / \quad x_0 = 0 \quad / \quad \gamma = 1$$

LT: a first example

Quadratic case ($\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_+^\star$)

Let us consider $f(x) = \alpha + \dfrac{1}{2}\beta(x - x_0)^2$

And look for the LT: $f^\star(t) \;=\; \sup_{x\in\mathbb{R}} \;[\, xt - f(x) \,]$

- Let us denote $g_t(x) = xt - f(x) = xt - \big(\alpha + \beta(x - x_0)^2 \,/\, 2\big)$
  - The derivative reads: $g_t'(x) = t - \beta(x - x_0)$
  - And the second derivative is: $g_t(x)'' = -\beta$
  - By nullification of $g_t'(x)$: $\bar{x} = x_0 + t/\beta$
  - Then by substitution: $f^\star(t) = g_t(\bar{x})$

$$f^\star(t) \;=\; \frac{1}{2\beta}t^2 + tx_0 - \alpha$$

- Have a look at the case $\alpha = 0$, $x_0 = 0$ and $\beta = 1\ldots$

LT: a generic result for explicitation (a)

A swiss army formula: Legendre formula

$$f^{\star}(t) \;=\; \sup_{x\in\mathbb{R}} \left[\, xt - f(x) \,\right]$$

- Let us denote $g_t(x) = xt - f(x)$
  - The derivative reads: $g_t'(x) = t - f'(x)$
  - And the second derivative is: $g_t(x)'' = -f''(x)$
  - By nullification of $g_t'(x)$:

$$\begin{aligned} t - f'(\bar{x}) &= 0 \\ \bar{x} &= f'^{-1}(t) \;=\; \chi(t) \end{aligned}$$

- Then by substitution:

$$f^{\star}(t) \;=\; g_t(\bar{x}) \;=\; t\bar{x} - f(\bar{x}) \;=\; t\chi(t) - f\left[\chi(t)\right]$$

LT: a generic result for explicitation (b)

Derivatives

- Convex conjugate made explicit

$$f^\star(t) = t\chi(t) - f\left[\chi(t)\right] \quad \text{with } \chi = f^{'-1}$$

- The derivative reads:

$$\begin{aligned} f^{\star'}(t) &= \chi(t) + t\chi'(t) - \chi'(t)\, f'\left[\chi(t)\right] \\ &= \chi(t) \\ &= f^{'-1}(t) \end{aligned}$$

- And the second derivative is:

$$f^{\star''}(t) = \chi(t)' = \frac{1}{f''\left[\chi(t)\right]} \quad > 0$$

  - Hence $f^\star$ is convex...
  - ...and in fact $f^\star$ is always convex...

LT: a key result

Double conjugate

$$f^{\star\star}(x) = f(x)$$

$$f^{\star\star}(t) \;=\; \sup\; [\,xt - f^{\star}(x)\,]$$

- Let us note $h_t(x) = xt - f^{\star}(x)$ and calculate the derivative:
$$h_t'(x) \;=\; t - f^{\star'}(x) \;=\; t - f^{'-1}(x) \;=\; t - \chi(x)$$

- Nullify the derivative:
$$t - \chi(\bar{x}) = 0$$

- By substitution
$$\begin{aligned} f^{\star\star}(t) \;=\; h_t(\bar{x}) &= \bar{x}t - f^{\star}(\bar{x}) \\ &= \bar{x}t - [\bar{x}\chi(\bar{x}) - f(\chi(\bar{x}))] \\ &= \bar{x}t - \bar{x}t + f(t) \\ &= f(t) \end{aligned}$$

Outcome for LT: “un théorème vivant”

Definition

Let us consider $f : \mathbb{R} \longrightarrow \mathbb{R}$

- strictly convex
- once (or twice) differentiable

$$f^\star(t) \;=\; \sup_{x\in\mathbb{R}} \left[\, xt - f(x) \,\right]$$

Properties

$$f^\star(t) = t\chi(t) - f\left[\chi(t)\right] \quad \text{with } \chi = f^{'^{-1}}$$

$$f^{\star'} = f^{'^{-1}} = \chi$$

$$f^{\star''}(t) = 1 \,/\, f^{''}\left[\chi(t)\right]$$

$$f^\star \text{ is convex}$$

$$\boxed{f^{\star\star}(x) = f(x)}$$

Half-quadratic enchantement (start repeated)

- Reminder of the criterion

$$\mathcal{J}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \sum_{p \sim q} \varphi(x_p - x_q)$$

- Minimisation based on quadratic
  - Original idea of [Geman + Yang, 95]
  - Set of auxiliary variables a_{pq} so that: $\varphi(\delta_{pq}) \quad \longleftrightarrow \quad \delta_{pq}^2$

$$\varphi(\delta) = \inf_a \left[\frac{1}{2}(\delta - a)^2 + \zeta(a) \right]$$

  - With appropriate $\zeta(a)$

- Extended criterion

$$\tilde{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{a}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \sum_{p \sim q} \frac{1}{2} \left[(x_p - x_q) - a_{pq} \right]^2 + \zeta(a_{pq})$$

und natürlich:

$$\mathcal{J}(\boldsymbol{x}) = \inf_{\boldsymbol{a}} \tilde{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{a})$$

A theorem in action: half-quadratic (beginning)

Problem statement

Consider a potential φ, convex or not, and look for ζ such that

$$\varphi(\delta) = \inf_{a \in \mathbb{R}} \left[(\delta - a)^2/2 + \zeta(a) \right]$$

- Let us define g such that it is strictly convex:

$$g(\delta) = \delta^2/2 - \varphi(\delta)$$

- Consider its LT:

$$\begin{aligned} g^\star(a) &= \sup_{\delta \in \mathbb{R}} \left[a\delta - g(\delta) \right] \\ &= \sup_{\delta \in \mathbb{R}} \left[\varphi(\delta) - (\delta - a)^2/2 \right] + a^2/2 \end{aligned}$$

- Let us set (reason explained on the next slide):

$$\zeta(a) = g^\star(a) - a^2/2 = \sup_{\delta \in \mathbb{R}} \left[\varphi(\delta) - (\delta - a)^2/2 \right]$$

A theorem in action: half-quadratic (middle)

- Take advantage of $g = g^{\star\star}$

$$\begin{aligned} g(\delta) &= g^{\star\star}(\delta) \\ \delta^2/2 - \varphi(\delta) &= \sup_a \, [\, a\delta - g^\star(\delta)\,] \end{aligned}$$

- Then:

$$\begin{aligned} \varphi(\delta) &= \delta^2/2 - \sup \, [a\delta - g^\star(\delta)] \\ &= \delta^2/2 + \inf \, [g^\star(\delta) - a\delta] \\ &= \delta^2/2 + \inf \, \left[\zeta(a) + a^2/2 - a\delta\right] \\ &= \inf \left[(\delta - a)^2/2 + \zeta(a)\right] \end{aligned}$$

- The icing on the cake, we have the minimiser:

$$\left[(\delta - a)^2/2 + \zeta(a)\right]' = (a - \delta) + \zeta'(a) = g^{\star'}(a) - \delta$$

then:

$$\bar{a} = g^{\star'^{-1}}(\delta) = g'(\delta) = \delta - \varphi'(\delta)$$

A theorem in action: half-quadratic (ending)

- Reminder: original criterion...

$$\mathcal{J}(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \sum_{p \sim q} \varphi(x_p - x_q)$$

- ...and extended criterion

$$\tilde{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{a}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \sum \frac{1}{2} \left[(x_p - x_q) - a_{pq}\right]^2 + \zeta(a_{pq})$$

- Algorithmic strategy: alternating minimisation

  1. Minimisation w.r.t. $\boldsymbol{x}$ for fixed $\boldsymbol{a}$: $\widetilde{\boldsymbol{x}}(\boldsymbol{a}) = \arg\min_{\boldsymbol{x}} \tilde{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{a})$
     Quadratic problem
  2. Minimisation w.r.t. $\boldsymbol{a}$ for fixed $\boldsymbol{x}$: $\widetilde{\boldsymbol{a}}(\boldsymbol{x}) = \arg\min_{\boldsymbol{a}} \tilde{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{a})$
     Separated and explicit update

- Remark:

  Non-quadratic with Interacting variables

$$\rightsquigarrow \begin{cases} \text{Interacting but simply quadratic} \\ \text{Non-quadratic but non-interacting} \end{cases}$$

Image update, given current auxiliary variables

- Non-separable but quadratic w.r.t. $\boldsymbol{x}$

$$\tilde{\mathcal{J}}(\boldsymbol{x}) \quad \# \quad \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \mu \sum_{p\sim q} \frac{1}{2} \left[(x_p - x_q) - a_{pq}\right]^2$$

$$= \quad \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2 + \bar{\mu}\, \|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{a}\|^2$$

- Image update: standard…

$$\widetilde{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \tilde{\mathcal{J}}(\boldsymbol{x})$$

$$(\boldsymbol{H}^{\mathrm{t}}\boldsymbol{H} + \bar{\mu}\boldsymbol{D}^{\mathrm{t}}\boldsymbol{D})\,\widetilde{\boldsymbol{x}} = \boldsymbol{H}^{\mathrm{t}}\boldsymbol{y} + \bar{\mu}\,\boldsymbol{D}^{\mathrm{t}}\boldsymbol{a}$$

$$\widetilde{\boldsymbol{x}} = (\boldsymbol{H}^{\mathrm{t}}\boldsymbol{H} + \mu\boldsymbol{D}^{\mathrm{t}}\boldsymbol{D})^{-1}\,(\boldsymbol{H}^{\mathrm{t}}\boldsymbol{y} + \bar{\mu}\,\boldsymbol{D}^{\mathrm{t}}\boldsymbol{a})$$

$$\mathring{\widehat{\boldsymbol{x}}} = (\boldsymbol{\Lambda}_h^{\dagger}\boldsymbol{\Lambda}_h + \mu\boldsymbol{\Lambda}_d^{\dagger}\boldsymbol{\Lambda}_d)^{-1}(\boldsymbol{\Lambda}_h^{\dagger}\mathring{\boldsymbol{y}} + \bar{\mu}\,\boldsymbol{\Lambda}_d^{\dagger}\mathring{\boldsymbol{a}})$$

$$\mathring{\widehat{x}}_n = \frac{\mathring{h}_n^{*}\mathring{y}_n + \bar{\mu}\,\mathring{d}_n^{*}\mathring{a}_n}{|\mathring{h}_n|^2 + \mu|\mathring{d}_n|^2} \qquad \text{for } n = 1, \ldots N$$

Object update: other possibilities

Various options and many relationships...

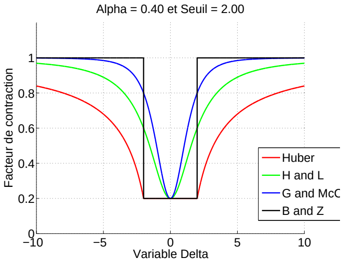
- Direct calculus, compact (closed) form, matrix inversion
- Algorithms for linear system
  - Gauss, Gauss-Jordan
  - Substitution
  - Triangularisation,...
- Numerical optimisation
  - gradient descent... and various modifications
  - Pixel wise, pixel by pixel
- Diagonalization
  - Circulant approximation and diagonalization by FFT
- Special algorithms, especially for 1D case
  - Recursive least squares
  - Kalman smoother or filter (and fast versions,...)

Auxiliary variables update, given current image

- Non quadratic but separable w.r.t. $\boldsymbol{a}$

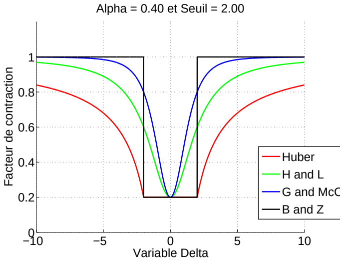
$$\tilde{\mathcal{J}}(\boldsymbol{a}) \# \sum_{p\sim q} \frac{1}{2}\left[(x_p - x_q) - a_{pq}\right]^2 + \zeta(a_{pq})$$

- Second enchantment:
  - Parallel computation (no loop): separability
  - Explicit (no inner-iterations): icing on the cake
- Update: $\widetilde{a}_{pq} = \delta_{pq} - \varphi'(\delta_{pq})$
  - Huber: $\widetilde{a}_{pq} = \delta_{pq}\left[1 - 2\alpha \min\left(1; s/\delta_{pq}\right)\right]$
  - Hyperbolic: $\widetilde{a}_{pq} = \delta_{pq}\left[\ldots\right]$



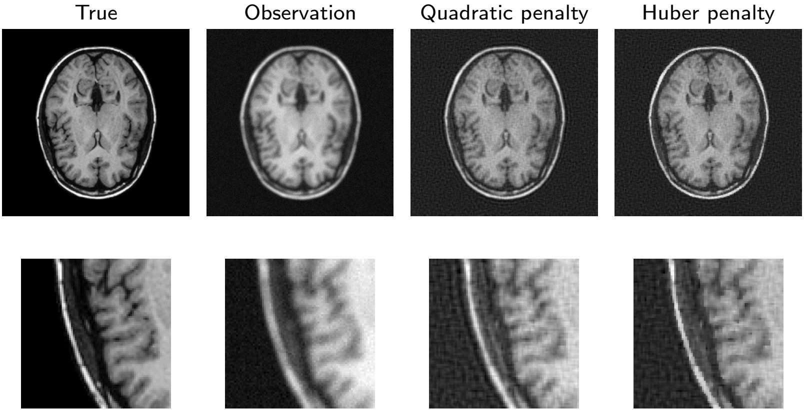
Auxiliary variables update, given current image

- Update: $\widetilde{a}_{pq} = \delta_{pq} - \varphi'(\delta_{pq})$
  - Blake und Zissermann: $\widetilde{a}_{pq} = \delta_{pq} [\ldots]$
  - $\ldots$: $\widetilde{a}_{pq} = \delta_{pq} [\ldots]$
  - Geman & McClure: $\widetilde{a} = \delta \left[1 - \frac{2\alpha}{(s/\delta)^2} \right]$



No more guarantees (my knowledge...):
existence, unicity... and convergence...

Result Huber



Conclusions

Synthesis

- Image deconvolution
- Edge preserving and non-quadratic penalties
  - Gradient of gray levels (and other transforms)
  - Convex (and differentiable) case and also some non-convex cases
- Numerical computations: half-quadratic approach
  - Iterative: quadratic $\oplus$ separable
    - Circulant case (diagonalization) $\rightsquigarrow$ FFT only
      (or numerical optimisation, system solvers,. . .)
    - Parallel (separable and explicit)

Extensions (next lectures)

- Also available for
  - non-invariant linear direct model
  - colour images, multispectral and hyperspectral
  - also signal, 3D and more, video, 3D+t. . .
- Including constraints $\rightsquigarrow$ better image resolution (next lecture)
- Hyperparameters estimation, instrument parameter estimation,. . .