

Image restoration

— **Convex approaches: penalties and constraints** —

An example in astronomy

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- Direct model and inverse problem
 - Interpolation-extrapolation / deconvolution / Fourier synthesis
 - Indetermination, non invertibility
- Prior information and regularized solution
 - Positivity and possible support
 - Point sources onto smooth background and double model
- Algorithmic aspects and numerical optimisation
- Data processing results
 - Simulated Data
 - NRH Data
- Conclusions et extensions

Interferometry: principles of measurement

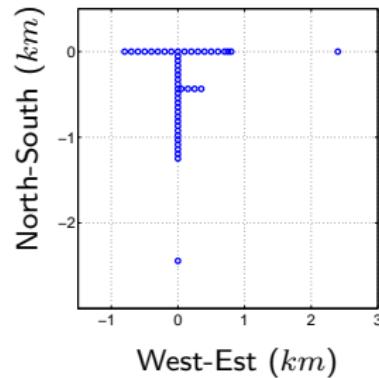
Physical principle [Thompson, Moran, Swenson, 2001]

- Antenna array \rightsquigarrow large aperture
- Frequency band, e.g., 164 MHz
- Couple of antennas interference \rightsquigarrow one measure in the Fourier plane

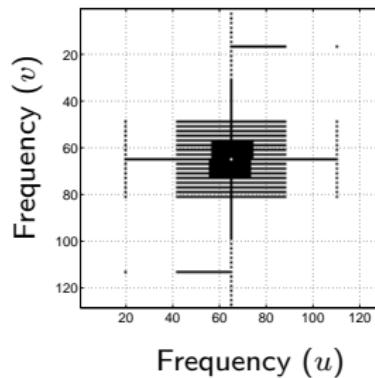
Picture site (NRH)



Antenna positions

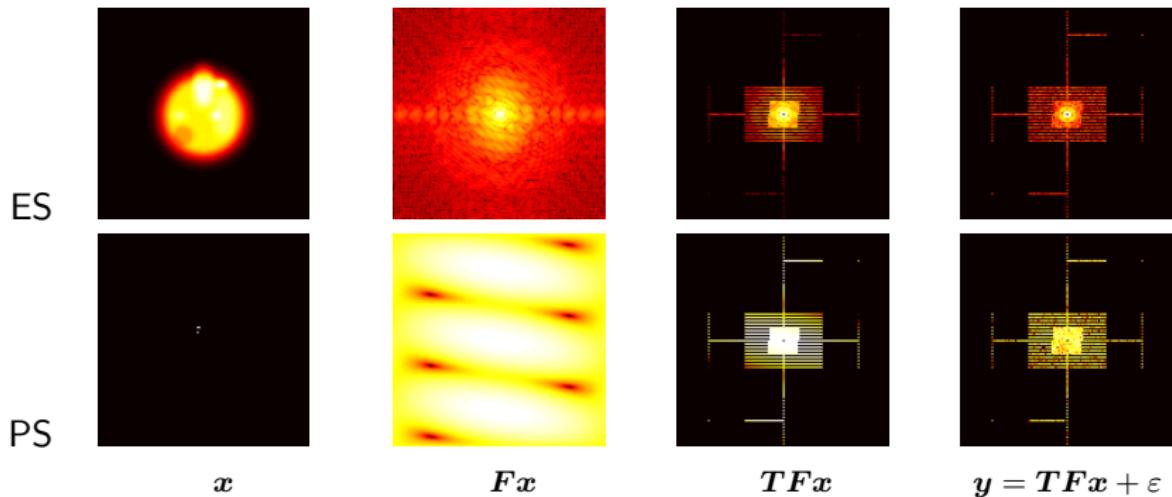


Fourier plane



- Knowledge of the sun, magnetic activity, eruptions, sunspots,...
- Forecast of sun events and their impact,...

Interferometer: example of measurements

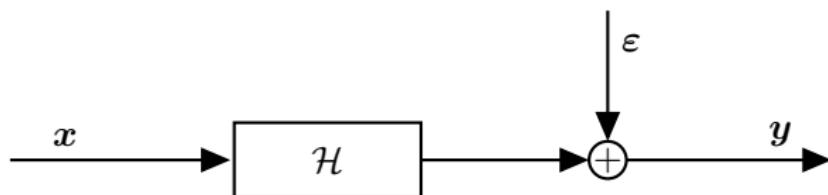


Instrument model

- Truncated and noisy Fourier transform

$$\mathbf{y} = \mathbf{T}\mathbf{F}\mathbf{x} + \boldsymbol{\varepsilon}$$

- $\mathbf{x} \in \mathbb{R}^N$: unknown image
- $\mathbf{y}, \boldsymbol{\varepsilon} \in \mathbb{C}^M$: measurements, errors
- \mathbf{F} : Fourier matrix ($N \times N$)
- \mathbf{T} : truncature matrix ($M \times N$), e.g., $\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$



- Difficulties: $M \ll N$, noise

Different formulations

- Fourier synthesis (original formulation)

$$\mathbf{y} = \mathbf{T}\mathbf{F}\mathbf{x} + \boldsymbol{\varepsilon}$$

- Interpolation – extrapolation: change of variable $\mathring{\mathbf{x}} = \mathbf{F}\mathbf{x}$

$$\mathbf{y} = \mathbf{T}\mathring{\mathbf{x}} + \boldsymbol{\varepsilon}$$

- Deconvolution: transformation of data

- $\mathring{\mathbf{y}} = \mathbf{F}^\dagger \mathbf{T}^t \mathbf{y}$
- $\mathbf{H} = \mathbf{F}^\dagger \mathbf{T}^t \mathbf{T} \mathbf{F}$

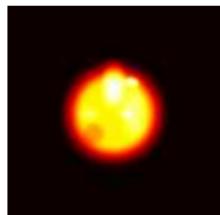
$$\mathring{\mathbf{y}} = \mathbf{H}\mathbf{x} + \widetilde{\boldsymbol{\varepsilon}}$$

- A few simple properties

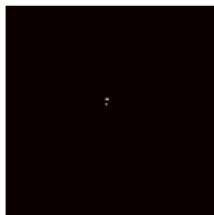
- $\mathbf{F}^\dagger \mathbf{F} = \mathbf{F} \mathbf{F}^\dagger = \mathbf{I}$: orthonormality
- \mathbf{T}^t : zero-padding matrix, $M \rightsquigarrow N$ (\mathbf{T}^t extends)
- $\mathbf{T}^t \mathbf{T}$: (diagonal) projection matrix, $N \rightsquigarrow N$ ($\mathbf{T}^t \mathbf{T}$ nullifies)
- $\mathbf{T} \mathbf{T}^t = \mathbf{I}_M$

Interferometry: illustration

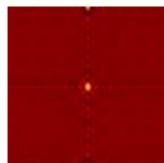
True map ES



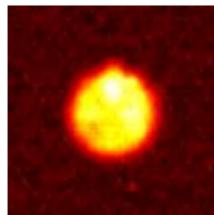
True map PS



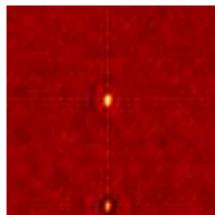
Dirty beam



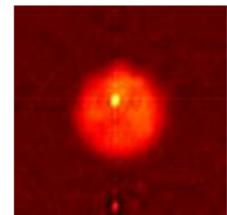
Dirty map ES



Dirty map PS



Dirty map PS + ES



Data based inversion and ill-posed character

- Deficient rank, missing data
 - $\mathbf{F}, \mathbf{T}, \mathbf{H}$: 1 singular value order M and 0 order $N - M$
- Infinity of Least-Squares solution
 - $\mathcal{J}_{\text{LS}}(\mathbf{x}) = \|\mathbf{y} - \mathbf{F}\mathbf{x}\|^2$
- Other solutions: minimum norm solution, TSVD, (quasi) Wiener ...
 - $\mathcal{J}_{\text{LS}}(\overset{\circ}{\mathbf{y}}) = 0$
 - $\mathcal{J}_{\text{LS}}\left(\overset{\circ}{\mathbf{y}} + [\mathbf{u} - \mathbf{F}^\dagger \mathbf{T}^t \mathbf{TFu}]\right) = 0, \text{ for all map } \mathbf{u}$
- Necessity of other information

Taking constraints into account: positivity and support

- Notation
 - \mathcal{M} : index set of the image pixels
 - $\mathcal{S}, \mathcal{D} \subset \mathcal{M}$: index set of a part of the image pixels

Investigated constraints here

- Positivity

$$C_p : \forall p \in \mathcal{M}, \quad x_p \geq 0$$

- Support

$$C_s : \forall p \in \bar{\mathcal{S}}, \quad x_p = 0$$

Extensions (non investigated here)

- Template

$$\forall p \in \mathcal{M}, \quad t_p^- \leq x_p \leq t_p^+$$

- Partially known map

$$\forall p \in \mathcal{D}, \quad x_p = m_p$$

Point sources + extended source

- Double-model [CIUCIU02, SAMSON03] et [MAGAIN98, PIRZKAL00]
 - $\boldsymbol{x} = \boldsymbol{x}_e + \boldsymbol{x}_p$
 - Direct model $\boldsymbol{y} = \boldsymbol{T}\boldsymbol{F}(\boldsymbol{x}_e + \boldsymbol{x}_p) + \boldsymbol{\varepsilon}$
 - New indeterminations
- Appropriate regularisation
 - $\mathcal{P}_e(\boldsymbol{x}_e) = \sum_{p \sim q} [x_e(p) - x_e(q)]^2$
 - $\mathcal{P}_p(\boldsymbol{x}_p) =$

Point sources + extended source

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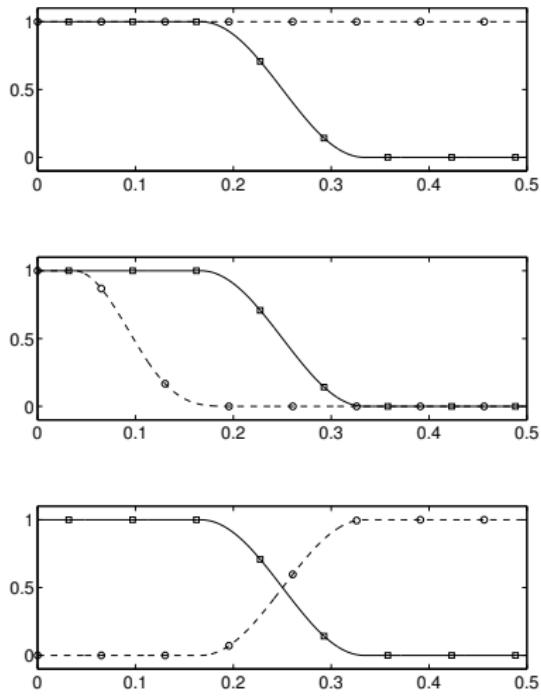
Point sources + extended source

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 - $\boldsymbol{x} = \boldsymbol{x}_e + \boldsymbol{x}_p$
 - Direct model $\boldsymbol{y} = \boldsymbol{T}\boldsymbol{F}(\boldsymbol{x}_e + \boldsymbol{x}_p) + \boldsymbol{\varepsilon}$
 - New indeterminations
- Appropriate regularisation

- $\mathcal{P}_e(\boldsymbol{x}_e) = \sum_{p \sim q} [x_e(p) - x_e(q)]^2$

- $\mathcal{P}_p(\boldsymbol{x}_p) = \sum |x_p(n)| = \sum x_p(n)$

Frequential analysis



Reduced frequency

Regularized criterion y regularized solution

- Criterion: penalized, quadratic, strictly convex

$$\mathcal{J}(\boldsymbol{x}_e, \boldsymbol{x}_p) = \|\boldsymbol{y} - \boldsymbol{T}\boldsymbol{F}(\boldsymbol{x}_e + \boldsymbol{x}_p)\|^2$$

$$+ \lambda_e \sum_{p \sim q} [x_e(p) - x_e(q)]^2 + \lambda_p \sum x_p(n)$$

$$+ \varepsilon_e \sum x_e(n)^2 + \varepsilon_p \sum x_p(n)^2$$

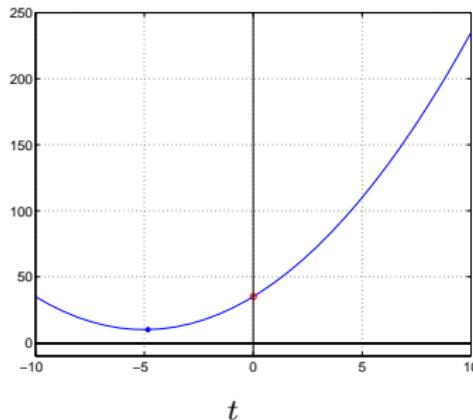
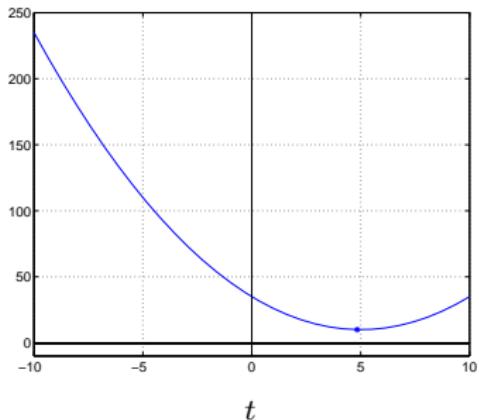
- Solution: unique constrained minimizer ($\boldsymbol{x} = [\boldsymbol{x}_e; \boldsymbol{x}_p]$)

$$(\widehat{\boldsymbol{x}}_e, \widehat{\boldsymbol{x}}_p) = \begin{cases} \arg \min \mathcal{J}(\boldsymbol{x}_e, \boldsymbol{x}_p) \\ \text{s.t. } (C) \end{cases} = \begin{cases} \arg \min \frac{1}{2} \boldsymbol{x}^\top \boldsymbol{Q} \boldsymbol{x} + \boldsymbol{q}^\top \boldsymbol{x} \\ \text{s.t. } \begin{cases} x_p = 0 & \text{for } p \in \bar{\mathcal{S}} \\ x_p \geq 0 & \text{for } p \in \mathcal{M} \end{cases} \end{cases}$$

Constraints: some illustrations

Positivity: one variable

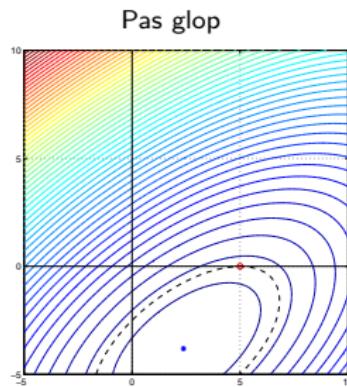
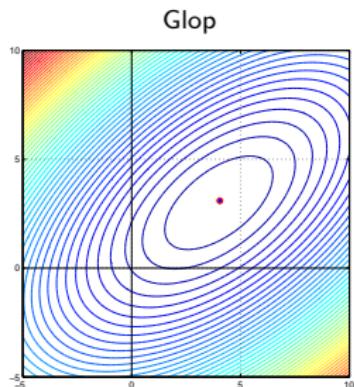
- One variable: $\alpha(t - \bar{t})^2 + \gamma$



- Non-constrained solution: $\hat{t} = \bar{t}$
- Constrained solution: $\hat{t} = \max [0, \bar{t}]$
- Active and inactive constraints

Positivity: two variables (1)

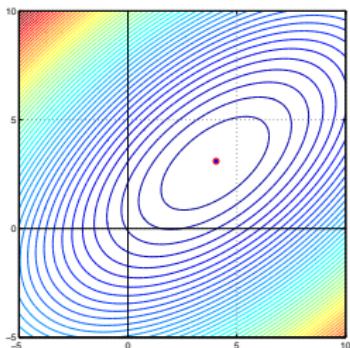
- Two variables: $\alpha_1(t_1 - \bar{t}_1)^2 + \alpha_2(t_2 - \bar{t}_2)^2 + \beta(t_2 - t_1)^2 + \gamma$



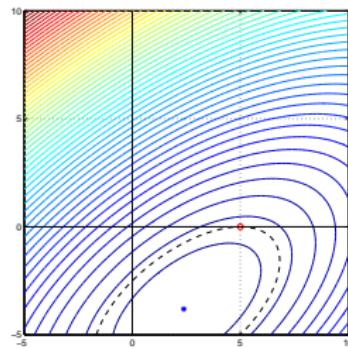
- Sometimes / often difficult to deduce the constrained minimiser from the non-constrained one

Positivity: two variables (2)

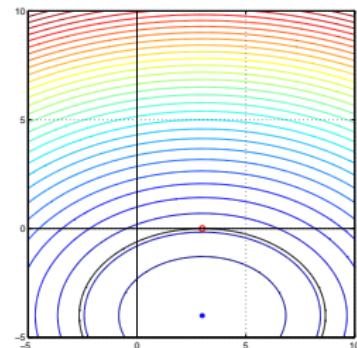
- Two variables: $\alpha_1(t_1 - \bar{t}_1)^2 + \alpha_2(t_2 - \bar{t}_2)^2 + \beta(t_2 - t_1)^2 + \gamma$



1



2a

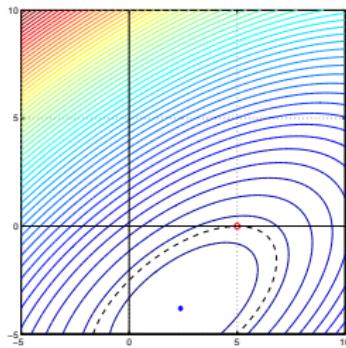


2b

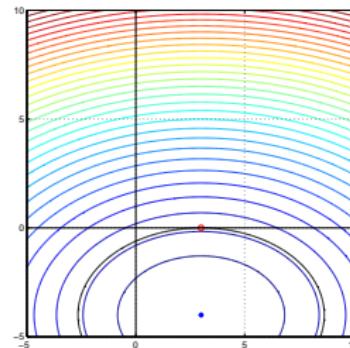
- Constrained solution = Non-constrained solution (1)
- Constrained solution \neq Non-constrained solution (2)
... so active constraints

Positivity: two variables (3)

- Two variables: $\alpha_1(t_1 - \bar{t}_1)^2 + \alpha_2(t_2 - \bar{t}_2)^2 + \beta(t_2 - t_1)^2 + \gamma$



2a



2b

- Constrained solution \neq Non-constrained solution (2)
... so active constraints
 - Constrained solution \neq Projected non-constrained solution (2a)
$$(\hat{t}_1; \hat{t}_2) \neq (\max [0, \bar{t}_1]; \max [0, \bar{t}_2])$$
 - Constrained solution = Projected non-constrained solution (2b)
$$(\hat{t}_1; \hat{t}_2) = (\max [0, \bar{t}_1]; \max [0, \bar{t}_2])$$

Problem

- Quadratic optimisation with linear constraints
- Difficulties
 - $N \sim 1\,000\,000$
 - Constraints \oplus non-separable variables

Existing algorithms

- Existing tools *with guaranteed convergence*
[BERTSEKAS 95,99; NOCEDAL 00,08; BOYD 04,11]
 - Gradient projection methods, constrained gradient method
 - Broyden-Fletcher-Goldfarb-Shanno (BFGS) and limited memory
 - Interior points and barrier
 - Pixel-wise descent
 - **Augmented Lagrangian, ADMM**
 - Constrained but separated + non-separated but non-constrained
 - Partial solutions still through FFT

Lagrangians und penalisation

- Equality constraint: $x_p = 0$

$$-\sum_{p \in \bar{\mathcal{S}}} \ell_p x_p + \frac{1}{2} c \sum_{p \in \bar{\mathcal{S}}} x_p^2$$

- Inequality constraint: $(x_p \geq 0) \rightsquigarrow (s_p - x_p = 0 ; s_p \geq 0)$

$$-\sum_{p \in \mathcal{S}} \ell_p (x_p - s_p) + \frac{1}{2} c \sum_{p \in \mathcal{S}} (x_p - s_p)^2$$

- Globally

$$\mathcal{L}(x, s, \ell) = \frac{1}{2} x^t Q x + q^t x - \ell^t (x - s) + \frac{1}{2} c (x - s)^t (x - s)$$

Iterative algorithm

$$\mathcal{L}(\mathbf{x}, \mathbf{s}, \boldsymbol{\ell}) = \frac{1}{2} \mathbf{x}^t \mathbf{Q} \mathbf{x} + \mathbf{q}^t \mathbf{x} - \boldsymbol{\ell}^t (\mathbf{x} - \mathbf{s}) + \frac{1}{2} c (\mathbf{x} - \mathbf{s})^t (\mathbf{x} - \mathbf{s})$$

- Iterate three steps

- ➊ Unconstrained minimization of \mathcal{L} w.r.t. \mathbf{x}

$$\tilde{\mathbf{x}} = -(\mathbf{Q} + c\mathbf{I})^{-1} (\mathbf{q} + [\boldsymbol{\ell} + cs]) \quad (\equiv \text{FFT})$$

- ➋ Minimization of \mathcal{L} w.r.t. \mathbf{s} , s.t. $s_p \geq 0$,

$$\tilde{s}_p = \begin{cases} \max(0, cx_p - \ell_p)/c & \text{for } p \in \mathcal{S} \\ 0 & \text{for } p \in \bar{\mathcal{S}} \end{cases}$$

- ➌ Update $\boldsymbol{\ell}$

$$\tilde{\ell}_p = \begin{cases} \max(0, \ell_p - cx_p) & \text{for } p \in \mathcal{S} \\ \ell_p - cx_p & \text{for } p \in \bar{\mathcal{S}} \end{cases}$$

Details about Q and q

- $q \rightsquigarrow$ dirty map:

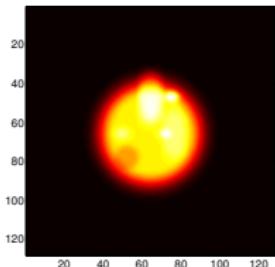
$$q = \left. \frac{\partial \mathcal{J}}{\partial \mathbf{x}} \right|_{x_e, x_p=0} = \begin{bmatrix} \frac{\partial \mathcal{J}}{\partial \mathbf{x}_e} \\ \frac{\partial \mathcal{J}}{\partial \mathbf{x}_p} \end{bmatrix} = -2 \begin{bmatrix} \overset{\circ}{\mathbf{y}} \\ \overset{\circ}{\mathbf{y}} + \lambda_c \mathbf{1}/2 \end{bmatrix}$$

- $Q \rightsquigarrow$ dirty beam:

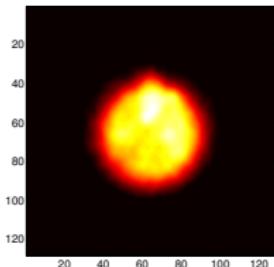
$$Q = \frac{\partial^2 \mathcal{J}}{\partial \mathbf{x}^2} = \begin{bmatrix} \frac{\partial^2 \mathcal{J}}{\partial \mathbf{x}_e^2} & \frac{\partial^2 \mathcal{J}}{\partial \mathbf{x}_e \partial \mathbf{x}_p} \\ \frac{\partial^2 \mathcal{J}}{\partial \mathbf{x}_p \partial \mathbf{x}_e} & \frac{\partial^2 \mathcal{J}}{\partial \mathbf{x}_p^2} \end{bmatrix} = \begin{bmatrix} \mathbf{H} + \lambda_s \mathbf{D}^t \mathbf{D} & \mathbf{H} \\ \mathbf{H} & \mathbf{H} + \varepsilon_s \mathbf{I} \end{bmatrix}$$

Simulated data results

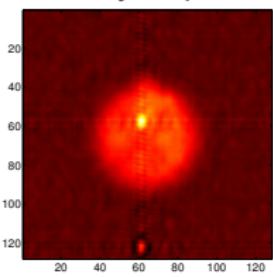
True extended object x_e^*



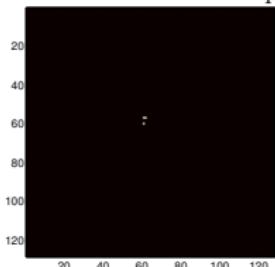
Estimated extended object \widehat{x}_e



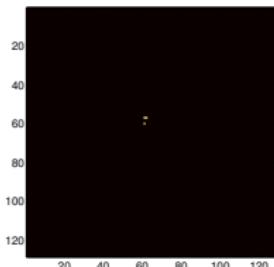
Dirty map



True point object x_p^*

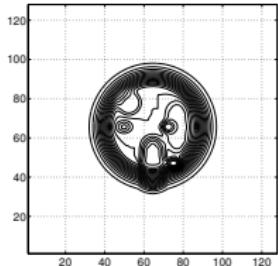


Estimated point object \widehat{x}_p

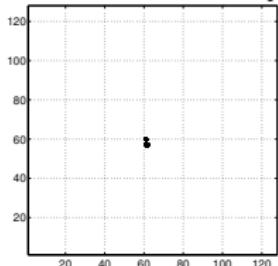


Simulated data results

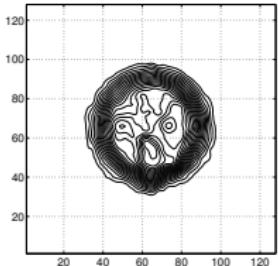
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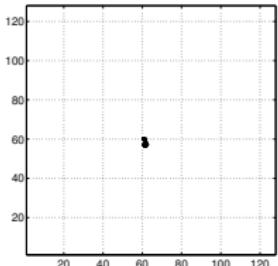
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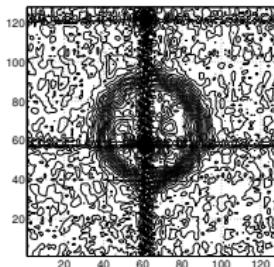
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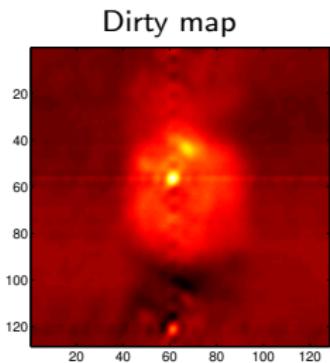
Estimated point object \hat{x}_p



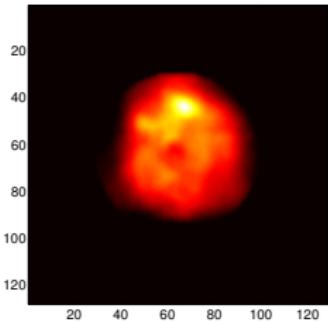
Dirty map



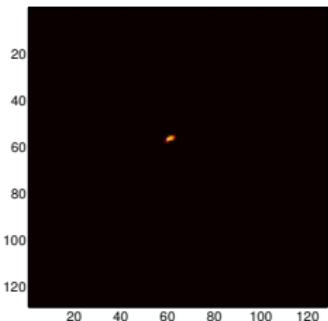
Real data: first results



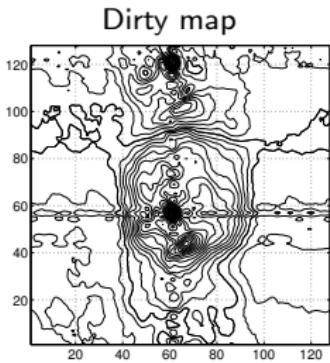
Estimated extended object \widehat{x}_e



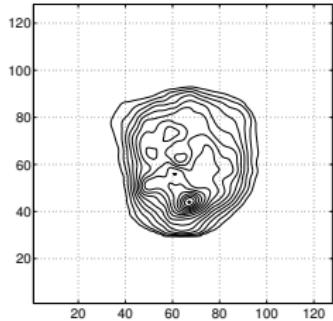
Estimated point object \widehat{x}_p



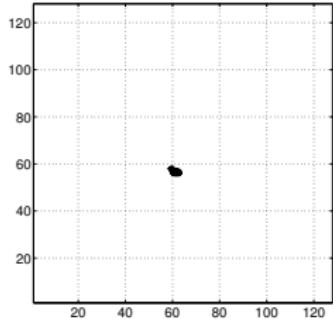
Real data: first results



Estimated extended object \widehat{x}_e



Estimated point object \widehat{x}_p



Conclusions

Synthesis

- Direct model and inverse problem
 - Interpolation-extrapolation/deconvolution/Fourier synthesis
 - Double-model and appropriate regularisation: point/background
 - Positivity and support
- Optimisation: lagrangians
- Simulations et real data: interferometry in radio-astronomy

Perspectives

- Quantitative assessment
- Case of a single map: an extended source / a set of point sources
- Non quadratic penalty \rightsquigarrow background resolution enhancement
- Data and/or sources “out grid”
- Hyperparameter estimation