Image restoration

— Introduction and examples —

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Topics

- Image restoration, deconvolution
    - Motivating examples: medical, astrophysical, industrial,. . .
    - Various problems: Fourier synthesis, deconvolution,. . .
    - Missing information: ill-posed character and regularisation

- Three types of regularised inversion

    1. Quadratic penalties and linear solutions
        - Closed-form expression
        - Computation through FFT
        - Numerical optimisation, gradient algorithm
    2. Non-quadratic penalties and edge preservation
        - Half-quadratic approaches, including computation through FFT
        - Numerical optimisation, gradient algorithm
    3. Constraints: positivity and support
        - Augmented Lagrangian and ADMM, including computation by FFT
        - Optimisation (*e.g.*, gradient), system solvers (*e.g.*, splitting)

- Bayesian strategy: a few incursions
    - Tuning hyperparameters, instrument parameters,. . .
    - Hidden / latent parameters, segmentation, detection,. . .

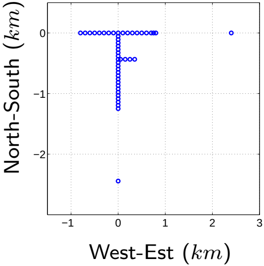
Interferometry: principles of measurement

Physical principle [Thompson, Moran, Swenson, 2001]

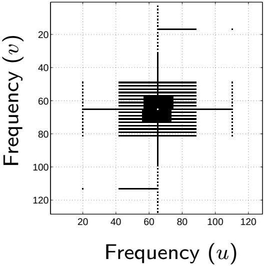
- Antenna array $\rightsquigarrow$ large aperture
- Frequency band, *e.g.*, 164 MHz
- Couple of antennas interference $\rightsquigarrow$ one measure in the Fourier plane

Picture site (NRH)

Antenna positions



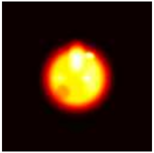
Fourier plane



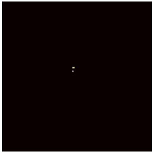
- Knowledge of the sun, magnetic activity, eruptions, sunspots,. . .
- Forecast of sun events and their impact,. . .

Interferometry: illustration

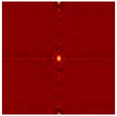
True map ES



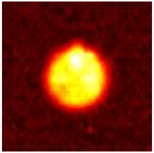
True map PS



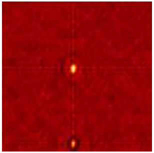
Dirty beam



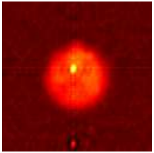
Dirty map ES



Dirty map PS



Dirty map PS + ES

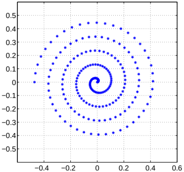


MRI: principles of measurement

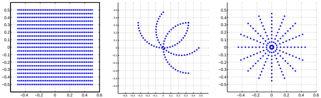
Physical principle [Alaux 92]

- High / Ultra-high magnetic field $\boldsymbol{B} \rightsquigarrow$ spin precession, $f \propto \|\boldsymbol{B}\|$
- Gradient $\boldsymbol{B} = \boldsymbol{B}_0 + B(x) \rightsquigarrow$ coding space - frequency
- Proton density $\rightsquigarrow$ signal amplitude
- Tissue, motion (physiological, perfusion, diffusion),...
- Sequence and system parameters (times, magnetic field,...)

GE Phantom



Frequency coverage



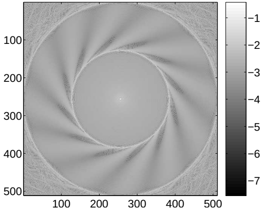
Other acquisition schemes

- Medical imaging, morphological and functional, neurology,...
- Fast MRI, cardiovascular applications, flow imaging,...

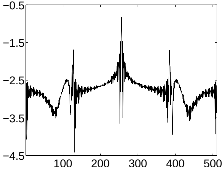
MRI: illustration

Phantom (true)

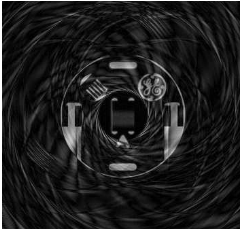
Instrument response



Cross-section

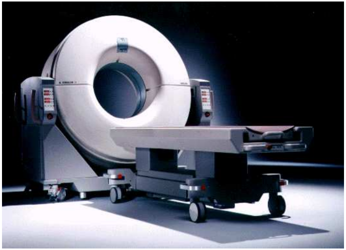


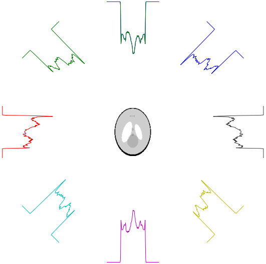
"Pseudo-observations"



Physical principle

- X-rays absorption $\rightsquigarrow$ radiography
- Rotation around the objet $\rightsquigarrow$ a set of radiographies (sinogram)
- Radon transform





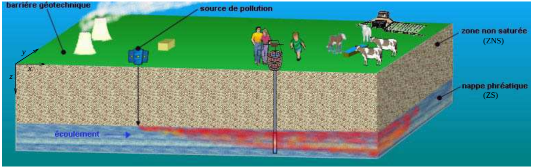
- Materials analysis and characterization, airport security,...
- Medical imaging: diagnosis, therapeutic follow-up,...

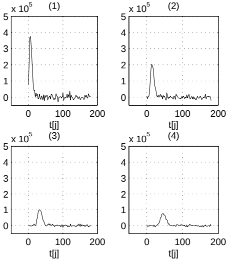
X-ray tomography: illustration

Hydrogeology and source identification

Physical principle

- Source: chemical, radioactive, odor,. . .
- Transport phenomena in porous media
- Groundwater sensors (drilling)

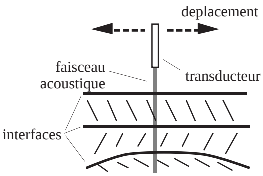


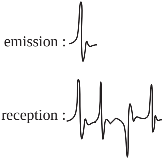


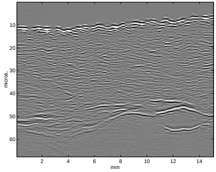
- Monitoring: electricity generation, chemical industry,. . .
- Knowledge for its own sake: subsoils, transportation, geology,. . .

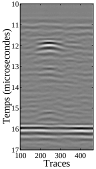
Physical principle

- Interaction: ultrasonic wave $\leftrightarrow$ medium of interest
- Acoustic impedance: inhomogeneity, discontinuity, medium change







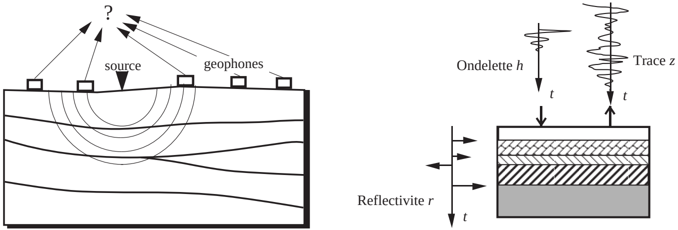


- Industrial control: very-early cracks detection (nuclear plants,. . .)
- Non destructive evaluation: aeronautics, aerospace,. . .
- Tissue characterisation, medical imaging,. . .

Seismic reflection method

Physical principle

- Interaction: mechanical wave $\leftrightarrow$ medium of interest
- Acoustic impedance: inhomogeneity, discontinuity, medium change

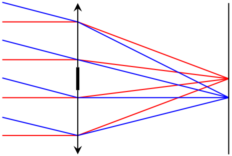


- Mineral and oil exploration,. . .
- Knowledge of subsoils and geology,. . .

Optical imaging (and infrared, thermography)

Physical principle

- Fundamentals of optics (geometrical and physical) $\rightsquigarrow$ stain
- CCD sensors or bolometers $\rightsquigarrow$ spatial and time response

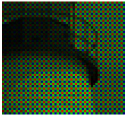


- Public space surveillance (car traffic, marine salvage,...)
- Satellite imaging: astronomy, remote sensing, environment
- Night vision, smokes / fogs / clouds, bad weather conditions

Digital photography and demosaicing

Physical principle

- Matrix / filter / Bayer mosaic: red, green, blue
- Chrominance and luminance

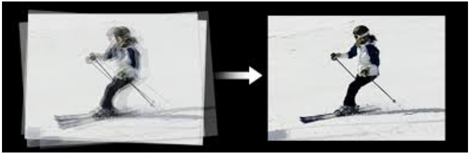


- Holiday pictures
- Surveillance (public space, car traffic,. . .)

The case of super-resolution

Physical principle

- Time series of images $\rightsquigarrow$ over-resolved images
- Sub-pixel motion $\sim$ over-sampling
- Motion estimation + Restoration



- Same applications. . . with higher resolution

And other imaging... fields, modalities, problems,...

Fields

- Astronomy, geology, hydrology,...
- Thermography, fluid mechanics, transport phenomena,...
- Medical: diagnosis, prognosis, theranostics,...
- Remote sensing, airborne imaging,...
- Surveillance, security,...
- Non destructive evaluation, control,...
- Computer vision, under bad conditions,...
- Augmented reality, computer vision & graphics,...
- Photography, games, recreational activities, leisures,...
- ...

$\rightsquigarrow$ Health, knowledge, leisure,...
$\rightsquigarrow$ Augmented Reality, Computer Vision & Graphics,...
$\rightsquigarrow$ Aerospace, aeronautics, transport, energy, industry,...

And other imaging... fields, modalities, problems,...

Modalities

- Interferometry (radio, optical, coherent,...)
- Magnetic Resonance Imaging
- Tomography based on X-ray, optical wavelength, tera-Hertz,...
- Ultrasonic imaging, sound, mechanical
- Holography
- Polarimetry: optical and other
- Synthetic aperture radars
- Microscopy, atomic force microscopy
- Camera, photography
- Lidar, radar, sonar,...
- ...

$\leadsto$ Essentially "wave $\leftrightarrow$ matter" interaction

And other imaging. . . fields, modalities, problems,. . .

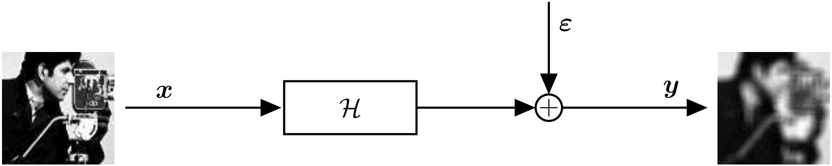
"Signal – Image" problems

- Denoising
- Edge / contrast enhancement
- Missing data
  - inpainting / interpolation
  - outpainting / extrapolation
- Deconvolution
- Inverse Radon
- Fourier synthesis
- . . .
- And also:
  - Segmentation
  - Detection of impulsions, salient points,. . .
  - . . .

$\rightsquigarrow$ In the following lectures: deconvolution-denoising

Inversion: standard question

$$\boldsymbol{y} = \mathcal{H}(\boldsymbol{x}) + \boldsymbol{\varepsilon} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{\varepsilon} = \boldsymbol{h} \star \boldsymbol{x} + \boldsymbol{\varepsilon}$$

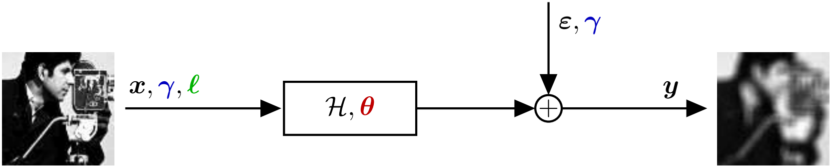


$$\widehat{\boldsymbol{x}} = \widehat{\mathcal{X}}(\boldsymbol{y})$$

Restoration, deconvolution-denoising

- General problem: ill-posed inverse problems, *i.e., lack of information*
- Methodology: regularisation, *i.e., information compensation*
  - Specificity of the inversion / reconstruction / restoration methods
  - Trade off and tuning parameters
- Limited quality results

$$\boldsymbol{y} = \mathcal{H}(\boldsymbol{x}) + \boldsymbol{\varepsilon} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{\varepsilon} = \boldsymbol{h} \star \boldsymbol{x} + \boldsymbol{\varepsilon}$$



$$\widehat{\boldsymbol{x}} = \widehat{\mathcal{X}}(\boldsymbol{y})$$

$$\left[\, \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\ell}} \,\right] = \widehat{\mathcal{X}}(\boldsymbol{y})$$

More estimation problems

- Hyperparameters, tuning parameters: *unsupervised*
- Instrument parameters (resp. response): *myopic* (resp. *blind*)
- Hidden variables: edges, regions, singular points,. . . : *augmented*
- Different models for image, noise, response,. . . : *model selection*

Issues and framework

Inverse problems

- Instrument model, direct / forward model
- Involves physical principles of
  - the phenomenom at stake
  - the acquisition system, the sensor
- Inverse
  - undo the degradations, surpass natural resolution
  - from consequences to causes
  - restore / rebuild / retrieve
- Ill-posed / ill-conditioned character and regularisation

Framework of this course

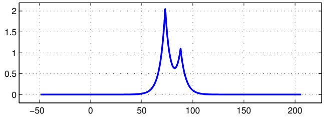
- Direct model
  - linear and shift invariant, *i.e.*, convolutive
  - including additive error (model and measurement)
- Regularisation through penalties and constraints
- Criterion optimisation and convexity

Some historical landmarks

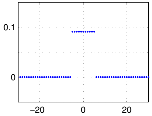
- Quadratic approaches and linear filtering ~ 60
  - Phillips, Twomey, Tikhonov
  - Kalman
  - Hunt (and Wiener ~ 40)
- Extension: discrete hidden variables ~ 80
  - Kormylo & Mendel (impulsions, peaks,...)
  - Geman & Geman (lines, contours, edges,...)
  - Besag, Graffigne, Descombes (regions, labels,...)
- Convex penalties (also hidden variables,...) ~ 90
  - $\mathrm{L}_2 - \mathrm{L}_1$, Huber, hyperbolic: Sauer, Blanc-Féraud, Idier...
  - ...et les POCS
  - L_1: Alliney-Ruzinsky, Taylor ~ 79, Yarlagadda ~ 85 ...
  - And... L_1-boom ~ 2005
- Back to more complex models ~ 2000
  - Unsupervised, myopic, semi-blind, blind
  - Stochastic sampling (MCMC, Metropolis-Hastings...)

Example due to Hunt (“square” response) [1970]

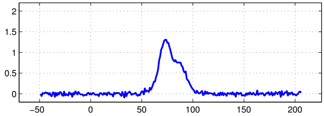
- Convolutive model: $\boldsymbol{y} = \boldsymbol{h} \star \boldsymbol{x} + \boldsymbol{\varepsilon}$
- Samples averaging



True $\boldsymbol{x}$



Response $\boldsymbol{h}$

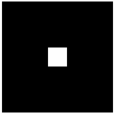


Observation $\boldsymbol{y}$

Example: photographed photographer (“square” response)

- Convolutive model: $\boldsymbol{y} = \boldsymbol{h} \star \boldsymbol{x} + \boldsymbol{\varepsilon}$
- Pixels averaging
- Think also about the Fourier domain

True $\boldsymbol{x}$

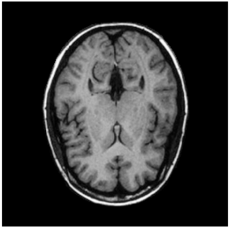


Spatial response $\boldsymbol{h}$

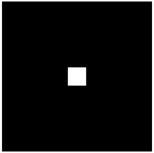
Observation $\boldsymbol{y}$

Example: brain ("square" response)

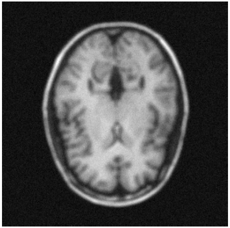
- Convolutive model: $\boldsymbol{y} = \boldsymbol{h} \star \boldsymbol{x} + \boldsymbol{\varepsilon}$
- Pixels averaging
- Think also about the Fourier domain



True $\boldsymbol{x}$

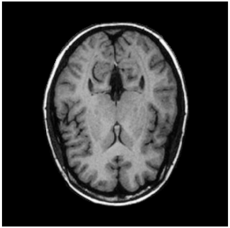


Spatial response $\boldsymbol{h}$

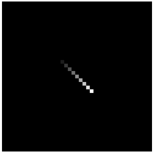


Observation $\boldsymbol{y}$

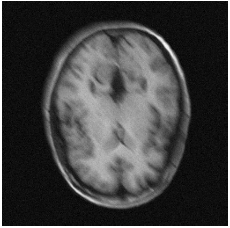
- Convolutive model: $\boldsymbol{y} = \boldsymbol{h} \star \boldsymbol{x} + \boldsymbol{\varepsilon}$
- Pixels averaging
- Think also about the Fourier domain



True $\boldsymbol{x}$



Spatial response $\boldsymbol{h}$



Observation $\boldsymbol{y}$

Convolution equation (discrete time / space)

- Examples of response

- Convolutive model

$$
\begin{aligned}
z(n) &= \sum_{p=-P}^{+P} h(p)\, x(n-p) \\
z(n,m) &= \sum_{p=-P}^{+P} \sum_{q=-Q}^{+Q} h(p,q)\, x(n-p, m-q)
\end{aligned}
$$

- Response: $h(p)$ or $h(p,q)$
  - impulse response, convolution kernel,...
  - ...point spread function, stain image

Convolution equation (discrete time, 1D): matrix form

- Linear $\rightsquigarrow$ matricial relation: $\boldsymbol{z} = \boldsymbol{H}\boldsymbol{x}$
- Shift invariance $\rightsquigarrow$ Tœplitz structure
- Short response $\rightsquigarrow$ band structure

$$
\begin{aligned}
& \vdots \\
z_{n-1} &= \\
z_n &= h_P x_{n-P} + \cdots + h_1 x_{n-1} + h_0 x_n + h_{-1} x_{n+1} + \cdots + h_{-P} x_{n+P} \\
z_{n+1} &= \\
& \vdots
\end{aligned}
$$

$$
\boldsymbol{H} = \begin{bmatrix}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & h_P & \dots & h_0 & \dots & h_{-P} & 0 & 0 & 0 & 0 & 0 & \\
\dots & 0 & h_P & \dots & h_0 & \dots & h_{-P} & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & h_P & \dots & h_0 & \dots & h_{-P} & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & h_P & \dots & h_0 & \dots & h_{-P} & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & h_P & \dots & h_0 & \dots & h_{-P} & 0 & \dots \\
 & 0 & 0 & 0 & 0 & 0 & h_P & \dots & h_0 & \dots & h_{-P} & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{bmatrix}
$$

See exercise regarding Tœplitz and circulant matrices. . .

Short and important incursion in “continuous statement“

- More realistic modelling of physical phenomenon
- Continuous variable convolution (1D, 2D and 3D,. . .)
- Observations
  - Sampling (discretization) of output
  - Finite number of samples
- Decomposition of unknown object
- Again “discrete” convolution

Convolution equation: continuous variable

Convolutive integral equation

$$
\begin{aligned}
z(t) &= \int x(\tau)\, h(t-\tau)\ \mathrm{d}\tau \\
z(u,v) &= \iint x(u',v')\, h(u-u',v-v')\ \mathrm{d}u'\mathrm{d}v' \\
z(u,v,w) &= \iiint x(u',v',w')\, h(u-u',v-v',w-w')\ \mathrm{d}u'\mathrm{d}v'\mathrm{d}w'
\end{aligned}
$$

More generally: Fredholm integral equation (first kind)

$$
\begin{aligned}
z(t) &= \int x(\tau)\, h(t,\tau)\ \mathrm{d}\tau \\
z(u,v) &= \iint x(u',v')\, h(u,u',v,y')\ \mathrm{d}u'\mathrm{d}v' \\
z(u,v,w) &= \iiint x(u',v',w')\, h(u,u',v,y',w,w')\ \mathrm{d}u'\mathrm{d}v'\mathrm{d}w'
\end{aligned}
$$

Continuous convolution and discrete observations

- Convolutive integral equation

$$\text{for } t \in \mathbb{R} \ : \qquad z(t) \ = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, \mathrm{d}\tau$$

- Measurement
  - Discrete data (just sampling, no approximation)...

$$z_n \ = \ z(nT_\mathrm{s}) \ = \int_{-\infty}^{+\infty} x(\tau)\, h(nT_\mathrm{s}-\tau)\, \mathrm{d}\tau$$

  - ...and finite number of data: $n = 1, 2, \ldots, N$.
  - Unknown object remains "continuous variable": $x(t)$, for $t \in \mathbb{R}$

Object "decomposition-recomposition"

- "General" decomposition of continuous time object

$$x(\tau) = \sum_k x_k \, \varphi_k(\tau)$$

  - Fourier series and finite time extend (finite duration)
  - Cardinal sine and finite bandwidth
  - Spline, wavelets, Gaussian kernel...
  - ...

- Infinite dimensional linear algebra
  - Hilbert spaces, Sobolev spaces...
  - Basis, representations...
  - Inner products, norms, projections...

Object "de / re - composition": example of finite bandwidth

- "General" decomposition of continuous time object

$$x(\tau) \;=\; \sum_k x_k\,\varphi_k(\tau)$$

- Case of shifted version of a basic function φ_0

$$\varphi_k(\tau) \;=\; \varphi_0(\tau - k\delta)$$

- Special case with cardinal sine

$$\varphi_0(\tau) = \mathrm{sinc}\big[t/\delta\big] \quad \text{with} \;\; \mathrm{sinc}\big[u\big] = \frac{\sin \pi u}{\pi u}$$

- That is the Shannon reconstruction formula

$$x(\tau) \;=\; \sum_{k\in\mathbb{Z}} x_k\,\varphi_0(\tau - k\delta) \;=\; \sum_{k\in\mathbb{Z}} x_k\,\mathrm{sinc}\Big[\frac{\tau - k\delta}{\delta}\Big]$$

  - . . . and there is no approximation, no error if
    - the signal is •
    - and with $x_k =$ •

Object "de / re - composition": example of finite bandwidth

- "General" decomposition of continuous time object

$$x(\tau) \;=\; \sum_k x_k\, \varphi_k(\tau)$$

- Case of shifted version of a basic function φ_0

$$\varphi_k(\tau) \;=\; \varphi_0(\tau - k\delta)$$

- Special case with cardinal sine

$$\varphi_0(\tau) = \mathrm{sinc}\big[t/\delta\big] \quad \text{with} \;\; \mathrm{sinc}\big[u\big] = \frac{\sin \pi u}{\pi u}$$

- That is the Shannon reconstruction formula

$$x(\tau) \;=\; \sum_{k\in\mathbb{Z}} x_k\, \varphi_0(\tau - k\delta) \;=\; \sum_{k\in\mathbb{Z}} x_k\, \mathrm{sinc}\Big[\frac{\tau - k\delta}{\delta}\Big]$$

  - . . . and there is no approximation, no error if
    - the signal is of finite bandwidth
    - and with $x_k = x(kT_\mathrm{s})$ with T_s •

Object "de / re - composition": example of finite bandwidth

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  - ...and there is no approximation, no error if
    - the signal is of finite bandwidth
    - and with $x_k = x(kT_\mathrm{s})$ with T_s small enough: $F_\mathrm{s} > 2F_\mathrm{M}$

Convolution: continuous $\rightsquigarrow$ discrete

- Given that (discrete observation at time nT_s)

$$z_n \;=\; \int_{-\infty}^{+\infty} x(\tau)\, h(nT_\mathrm{s} - \tau)\; \mathrm{d}\tau \qquad \text{for} \;\; n = 1, 2, \ldots, N$$

- and that (case of shifted version of a basic function φ_0)

$$x(\tau) \;=\; \sum_k x_k\, \varphi_0(\tau - k\delta) \qquad \text{for} \;\; \tau \in \mathbb{R}$$

- By substitution, we have

$$\begin{aligned}
z_n \;&=\; \int_{-\infty}^{+\infty} \left[\sum_k x_k\, \varphi_0(\tau - k\delta) \right] h(nT_\mathrm{s} - \tau)\; \mathrm{d}\tau \\
&=\; \sum_k x_k \int_{-\infty}^{+\infty} \varphi_0(\tau - k\delta)\, h(nT_\mathrm{s} - \tau)\; \mathrm{d}\tau \\
&=\; \sum_k x_k \underbrace{\int_{-\infty}^{+\infty} \varphi_0(\tau)\, h([nT_\mathrm{s} - k\delta] - \tau)\; \mathrm{d}\tau}_{}
\end{aligned}$$

Convolution: continuous $\rightsquigarrow$ discrete

- Let us denote: $\bar{h} = \varphi_0 \star h$

$$\bar{h}(u) \quad = \quad \int_{-\infty}^{+\infty} \varphi_0(\tau)\, h(u-\tau)\ \mathrm{d}\tau$$

- We then have

$$\begin{aligned} z_n \quad &= \quad \sum_k x_k \int_{-\infty}^{+\infty} \varphi_0(\tau)\, h([nT_\mathrm{s} - k\delta] - \tau)\ \mathrm{d}\tau \\ &= \quad \sum_k x_k\, \bar{h}(nT_\mathrm{s} - k\delta) \end{aligned}$$

  - The z_n are given as a function of the x_k
  - It is a “discrete linear” transform
  - There is no apprximation

Convolution: continuous $\rightsquigarrow$ discrete

- A specific case when $\delta = T_\mathrm{s}/K$

$$\begin{aligned} z_n &= \sum_k x_k \, \bar{h}\,[nT_\mathrm{s} - k\delta] \\ &= \sum_k x_k \, \bar{h}\,[nK\delta - k\delta] \\ &= \sum_k x_k \, \bar{h}\,[(nK - k)\delta] \end{aligned}$$

  - Subsampled discrete convolution

- A specific case when $\delta = T_\mathrm{s}$, *i.e.*, $K = 1$

$$z_n = \sum_k x_k \, \bar{h}\,[(n - k)\delta]$$

  - A standard discrete convolution